ГОДИШНИК НА УНИВЕРСИТЕТА ПО АРХИТЕКТУРА, СТРОИТЕЛСТВО И ГЕОДЕЗИЯ
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A 2-DIMENSONAL GENERALIZATION OF ROLLE'S THEOREM

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ABSTRACT

A generalization in dimension 2 of the classical Rolle's Theorem is given in the present article. Namely, if $\varphi : \mathbb{S}^1 \to \mathbb{R}^1$ is a smooth function with m local and s absolute extrema ($m \geq s$) such that $s > \frac{m}{2} + 1$, then any smooth extension $f : \mathbb{D}^2 \to \mathbb{R}^1$ of φ has a critical point, i.e. a point $p \in \mathbb{D}^2$, where $\nabla f(p) = 0$.

It turns out that the contrary is also valid: if $\varphi : \mathbb{S}^1 \to \mathbb{R}^1$ is a smooth function with m local and s absolute extrema such that $s \leq \frac{m}{2} + 1$, then there is an extension $f : \mathbb{D}^2 \to \mathbb{R}^1$ of φ without critical points.

1 Main results

The well-known classical Rolle's Theorem has many different generalizations in various situations, such as - greater dimensions, complex or vector functions, etc. (see [1], [2], [3], [4]). Here we prove another generalization in dimension 2 that seems quite natural and, more or less, elementary. In brief, given a smooth function φ on the unit circle $\mathbb{S}^1$, we are trying to answer the question: What properties should possess the function φ so that any its smooth extension on the unit disk $\mathbb{D}^2$ will have a critical point (i.e. a point where both partial dervatives vanish)? Note that this situation is a straightforward generalization of the classical one-dimensional Rolle's Theorem. One obvious answer is $\varphi \equiv const$, but it turns out that there are many other functions with the desired property.

We shall prove in this note a sufficient condition (for 'good' functions φ) that is very easily checkable and which turns out to be also a necessary condition. This is namely the following

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Theorem 1. *Let $\varphi : \mathbb{S}^1 \to \mathbb{R}^1$ be a smooth function with m local and s absolute extrema $(m \geq s)$. Suppose that*

$$s > \frac{m}{2} + 1.$$

Then any smooth extension $f : \mathbb{D}^2 \to \mathbb{R}^1$ of φ has a critical point, i.e. a point $p \in \mathbb{D}^2$, where $\nabla f(p) = 0$.

(Here $\nabla f = (f_x, f_y)$ is the gradient of f.)

Broadly speaking, the majority of the local extrema of the function φ have to be absolute ones. Comments and examples are given at the end.

Proof. Suppose the contrary, i.e. that there exists a smooth function $f : \mathbb{D}^2 \to \mathbb{R}^1$ such that $f|_{\mathbb{S}^1} = \varphi$ and f has no critical points on $\mathbb{D}^2$. It is clear that the absolute extrema of φ are absolute extrema of f as well, since otherwise one finds a critical point of f inside $\mathbb{D}^2$. Let $a \in \mathbb{S}^1$ be a point of absolute maximum of φ. Consider on $\mathbb{S}^1$ the vector field $\mathbf{n}(x) = x$ (the exterior normal). It is easy to see that the function $\arg \nabla f - \arg \mathbf{n}$ changes sign in a from '+' to '-'. Similarly, if $a \in \mathbb{S}^1$ is a point of absolute maximum of φ, then $\arg \nabla f + \arg \mathbf{n}$ changes sign in a from '+' to '-' as well. On the other hand, if a is a critical point of φ, but not a local extremum, then neither $\arg \nabla f - \arg \mathbf{n}$, nor $\arg \nabla f + \arg \mathbf{n}$ changes sign in a. This is evident from the following reasoning: Let $a \in \mathbb{S}^1$, for example, be a point of absolute maximum of φ, then $\nabla f = \lambda \mathbf{n}$ for some $\lambda > 0$ and $f_x \dot{x} + f_y \dot{y} = 0$, where $(\dot{x}, \dot{y})$ is the tangent to $\mathbb{S}^1$. Let $b \in \mathbb{S}^1$ be a point preceding a and close to it. Then $f_x \dot{x} + f_y \dot{y} > 0$ in b, as φ is increasing in b. But noticing that the normal $\mathbf{n}$ is $\mathbf{n}(\dot{y}, -\dot{x})$, the above inequality implies that the mixed product $(\mathbf{n}, \nabla f, \mathbf{k}) > 0$, where $\mathbf{k}(0, 0, 1)$. Similarly, if $c \in \mathbb{S}^1$ is close to a and subsequent to it we get $(\mathbf{n}, \nabla f, \mathbf{k}) < 0$ in c. But these two inequalities about the mixed product imply that the function $\arg \nabla f - \arg \mathbf{n}$ changes sign in a from '+' to '-'. The case of absolute minimum is treated analogically. If K and L are closed curves in the torus $\mathbb{S}^1 \times \mathbb{S}^1$, by $K \circ L$ we denote their intersection index. If v is a unit field on $\mathbb{S}^1$, by Γ_v we denote its graph in the torus $\mathbb{S}^1 \times \mathbb{S}^1$. Consider now on $\mathbb{S}^1$ the unit vector field $V = \nabla f / \|\nabla f\|$ and define the invariant

$$\delta(V) = \Gamma_V \circ \Gamma_{\mathbf{n}} + \Gamma_V \circ \Gamma_{-\mathbf{n}}.$$

We shall prove that $\delta(V) > 2$. The above reasoning about the arguments implies that the s absolute extrema give contribution $+s$ in $\delta(V)$. But the condition $s > \frac{m}{2} + 1$ is equivalent to $s - (m - s) > 2$ and since the contribution of the $m - s$ non absolute extrema is at least $-(m - s)$, we get $\delta(V) > 2$. Let us note now that if V is homotopic to V', then $\delta(V) = \delta(V')$. The vector field $V|_{\mathbb{S}^1}$ is homotopic to a constant, as it extends to a non-zero field V on $\mathbb{D}^2$. Take the constant field $V_0 = (0, 1)$, then it is easy to compute $\delta(V_0) = 2$. Therefore $\delta(V) = \delta(V_0) = 2$, but this contradicts the inequality $\delta(V) > 2$ proven above. The theorem is proved. □

It turns out that the contrary assertion to Theorem 1 is also valid:

Theorem 2. *Let $\varphi : \mathbb{S}^1 \to \mathbb{R}^1$ be a smooth function with m local and s absolute extrema $(m \geq s)$ such that*

$$s \leq \frac{m}{2} + 1.$$

Then there is an extension $f : \mathbb{D}^2 \to \mathbb{R}^1$ of φ without critical points.

We shall give two alternative methods for the proof of this proposition.

First method: Making use of the inequality $s \leq \frac{m}{2} + 1$ we may find an unit vector field v on $\mathbb{S}^1$ such that $v.\tau = \varphi'$ and moreover, the degree of v is zero: $\deg v = 0$. Here $\tau = (\dot{x}, \dot{y})$ is the tangent to $\mathbb{S}^1$ and φ' is the derivative with respect to the arclength parameter. Since v is of zero degree, it extends on $\mathbb{D}^2$ to a nonsingular vector field V. Now, as $\int v.\tau ds = 0$, we may suppose that the field V is a *gradient* field: $V = \nabla F$ for some function $F : \mathbb{D}^2 \to \mathbb{R}^1$. (Hence F has no critical points in $\mathbb{D}^2$ as the field V is nonsingular.) Consider now the restriction of F on $\mathbb{S}^1$, $\psi(s) = F(x(s), y(s))$. Then

$$\psi' = F_x \dot{x} + F_y \dot{y} = v.\tau = \varphi'.$$

Therefore $\psi = \varphi + C$, in other words $F|_{\mathbb{S}^1} = \varphi + C$. But then $f = F - C$ is an extension of φ without critical points.

Second method: Consider the class $\mathcal{F}$ of all smooth immersions of $\mathbb{S}^1$ in $\mathbb{R}^2$ that may be extended to an immersion of $\mathbb{D}^2$ in $\mathbb{R}^2$. Now, the inequality $s \leq \frac{m}{2} + 1$ allows us to find a smooth $\phi : \mathbb{S}^1 \to \mathbb{R}^1$ such that $\lambda = (\phi, \varphi)$ is an immersion of class $\mathcal{F}$. Let $\Lambda : \mathbb{D}^2 \to \mathbb{R}^2$ be an immersion extending λ. Then writing $\Lambda = (\Lambda_1, \Lambda_2)$, it is clear that Λ_2 is an extension of φ without critical points.

Let us notice that the class $\mathcal{F}$ has no simple description, although it may be resolved by different algorithms (c.f. [5])

Final remarks.

1. The unit circle $\mathbb{S}^1$ may be replaced by any smooth simple closed curve in $\mathbb{R}^2$ and the proofs remain (almost) the same.

2. The case $s \leq \frac{m}{2} + 1$ may have many different (topologically non equivalent) solutions, so it may become a source of further investigations.

3. As in the classical case, Rolle's Theorem generalizes to Lagrange's Theorem, in the sense that given a vector ξ, we may describe the class of all smooth functions $\varphi : \mathbb{S}^1 \to \mathbb{R}^1$ such that for any its smooth extension $f : \mathbb{D}^2 \to \mathbb{R}^1$ there is $p \in \mathbb{D}^2$, so that $\nabla f(p) = \xi$.

4. Of course, the proof of Theorem 2 is not complete, the formal proof will be given in a subsequent publication.

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ОБОБЩЕНИЕ НА ТЕОРЕМАТА НА РОЛ В РАЗМЕРНОСТ 2

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Ключови думи: *теорема на Рол, критична точка, индекс*

РЕЗЮМЕ

Получено е обобщение в размерност 2 на класическата теорема на Рол. По-точно, ако $\varphi : \mathbb{S}^1 \to \mathbb{R}^1$ е гладка функция с m локални и s абсолютни екстремуми $(m \geq s)$, такава че $s > \frac{m}{2} + 1$, то всяко гладко продължение $f : \mathbb{D}^2 \to \mathbb{R}^1$ на φ има критична точка, т.е. точка $p \in \mathbb{D}^2$, където $\nabla f(p) = 0$.

Оказва се, че обратното също е вярно: ако $\varphi : \mathbb{S}^1 \to \mathbb{R}^1$ е гладка функция с m локални и s абсолютни екстремуми, такава че $s \leq \frac{m}{2} + 1$, то съществува продължение $f : \mathbb{D}^2 \to \mathbb{R}^1$ на φ без критични точки.