ГОДИШНИК НА УНИВЕРСИТЕТА ПО АРХИТЕКТУРА, СТРОИТЕЛСТВО И ГЕОДЕЗИЯ
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BROUWER FOLIATIONS IN NON-RECURRENT DOMAINS

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ABSTRACT

Orientation preserving homeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ with a finite set P of fixed points are considered. It is shown that if the restriction $f|_{\mathbb{R}^2\setminus P}$ has no recurrent points with respect to some metrics, then $\mathbb{R}^2\setminus P$ admits a proper 1-dimensional foliation $\mathcal{F}$ such that each leaf S is free (i.e. is disjoint from its image) and together with $f(S)$ is bounding a free domain (Theorem 1). Moreover, in case f is a free map, then each leaf of $\mathcal{F}$ defines a *translation domain* (Remark 2). Non recurrent and free dynamics is available, for example, if the sum of the indices of any subset of P does not equal 1 (Corollary 1).

Furthermore, it is shown that if p_0 is the unique fixed point with topological index $\neq 1$ of an orientation preserving local homeomorphism $f : U \to \mathbb{R}^2$, then there is a foliation of U with good dynamical properties. Namely, for any extension $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ of f with unique fixed point p_0, there exists a foliation of $\mathbb{R}^2$ with proper lines, such that any leaf is disjoint from its image and is generating a translation domain.

1 Introduction

The two dimensional dynamics is a fast developing area of topological dynamics which made recently a significant progress (c.f. [1],[2],[3],[4]). A central result here is the famous Brouwer's Plane Translation Theorem:

Theorem (Brouwer [5]). *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving self-homeomorphism of the plane without fixed points. Then each point $x_0 \in \mathbb{R}^2$ is contained in a properly embedded line l such that $l \cap f(l) = \emptyset$ and l is separating $f(l)$ from $f^{-1}(l)$.*

This theorem is of big importance for the dynamics of planar homeomorphisms, as it is easy to see that it implies that each point x_0 is contained in an open invariant domain, where f is topologically conjugated to the canonical translation $(x, y) \to (x+1, y)$ of the plane.

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Brouwer's Plane Translation Theorem has different proofs (c.f. [4],[6],[7]), although the first attempts were incorrect or even erroneous (see the overview in [4]).

Recently Patrice Le Calvez [8] showed that not only through any point $x_0 \in \mathbb{R}^2$ passes a Brouwer line, but there is a foliation of the plane consisting of such lines. This is much stronger than Brouwer's Plane Translation Theorem, as we obtain in some sense a global control on the dynamics of f.

We give in this article an exposition of some of the author's results generalizing Le Calvez's theorem for self-homeomorphisms of the plane with finite number of fixed points P and whose restriction to $\mathbb{R}^2\backslash P$ has no *chain recurrent* points (the definition is given below) with respect to some metrics in $\mathbb{R}^2\backslash P$. It turns out that in this case the set $\mathbb{R}^2\backslash P$ admits a foliation of Brouwer 1-manifolds, where by *Brouwer 1-manifold S* we mean either a Brouwer line (closed in $\mathbb{R}^2\backslash P$), or a *Brouwer circle*, (i.e. a topological circle) such that S is *free* (S is disjoint from its image) and together with $f(S)$ is bounding a free domain. For fixed points free homeomorphisms of the plane Brouwer circles are impossible, as follows from Brouwer's Fixed Point Theorem. If f is fixed points free, it is easy to see that each Brouwer line l is separating $f^{-1}(l)$ from $f(l)$. So, we get a generalization of Le Calvez's theorem, since the chain recurrent set of a fixed points free and orientation preserving homeomorphism of the plane $\mathbb{R}^2$ is empty with respect to some metrics, as follows from the famous Franks' Lemma [1].

We further apply this result to the study of the dynamics of a planar homeomorphism around an isolated fixed point p_0 of index $\neq 1$. This is a classical problem of particular interest (c.f. [9],[10],[11],[12]) which is difficult to be given a definitive answer, as this dynamics may be very complex, for example it may not even exist an invariant foliation for f in any neighborhood of the fixed point. Anyway, we show that there is a *Brouwer foliation* around p_0. More precisely, if U is a neighborhood of p_0, not containing other fixed points of f, there exists a fixed points free extension $\widetilde{f}$ of $f|_U$ to all of $\mathbb{R}^2$ which admits a foliation of $\mathbb{R}^2\backslash\{p_0\}$ such that each leaf l of this foliation is a proper topological line in $\mathbb{R}^2\backslash\{p_0\}$ non intersecting its image and l is generating a translation domain for $\widetilde{f}$. The trace of this foliation on U may be considered as a local Brouwer foliation for f, which reflects, in some sense, the most important properties of the local dynamics around p_0.

For example, by means of this local foliation it is not difficult to compute the index of p_0. It is possible also to better visualize the dynamics around p_0 as the germs of planar foliations around an isolated singularity are completely classified by W.Kaplan in [13]. Of course, it doesn't mean at all that we get in such a way a classification of germs of planar homeomorphisms around an isolated fixed point. Among other peculiarities, it is possible for example an infinite number of non equivalent germs to have one and the same simple invariant foliation (say, a saddle point foliation) as described in [14]. Then, of course, the orthogonal foliation is a local Brouwer foliation for all of these germs. The problem of topological classification of germs of planar homeomorphisms around an isolated fixed point is considered very hard or even hopeless.

Let us note the interesting paper of Frederick le Roux [12] where he describes geometrically the dynamics around an isolated fixed point of index $\neq 1$ by means of "hyperbolic" and "elliptic" sectors. There is some analogy between his geometrical visualization and the Brouwer foliation constructed here.

The Brouwer foliation is far from being canonical, as simple examples show. So, one arrives to the following general problem:

What is the relationship between two local Brouwer foliation of one and the same homeomorphism around an isolated fixed point?

Let us notice that Patrick Le Calvez [15] has proved an important equivariant version of

Brouwer's Translation Theorem, which implies that for any fixed points free authomorphism of a 2-surface there is a foliation of the universal cover by Brouwer lines (with respect to the lifted homeomorphism), which is invariant with respect to the group of deck transformations. In such a way this foliation defines a foliation of the surface with free leafs (each one is disjoint from its image). Anyway, simple examples show that these leafs may not be proper, i.e. closed subsets of the surface, they may be even dense in the surface. So, our result is not covered by Le Calvez's theorem.

2 Definitions and statement of results

Let (X, ρ) be a metric space and $f : X \to X$ be a self-homeomorphism. An ε-chain in X is a finite sequence of points $x_1, x_2, ..., x_n$ such that

$$\rho(f(x_i), x_{i+1}) < \varepsilon,\ i = 1, ..., n-1.$$

We say that this is an ε-chain from x_1 to x_n. The ε-chain is a *cycle*, if $x_1 = x_n$. As usual, the *chain-recurrent* set of f is defined as

$$CR_\rho(f) = \{x \in X |\ \text{for any } \varepsilon > 0 \text{ there is an } \varepsilon\text{-cycle from } x \text{ to } x.\}$$

Let $l \subset X$ be a subset of X homeomorphic to R^1, then l is said to be a *proper line*, if it is a closed subset of X. A 1-dimensional foliation of X is *proper*, if all of its leafs are closed in X.

A subset $A \subset X$ is called *free*, if $f(A) \cap A = \emptyset$.

If p_0 is an isolated fixed point of f, by $Index(p_0)$ we denote the topological index of p_0 (see [18]).

Definition 1. *Let S be a closed subset of X. We shall say that S is generating a translation domain for f, if there is an embedding $\psi : S \times \mathbb{R} \to X$ such that*
1) $\psi(S \times \{0\}) = S$
2) $f(\psi(x)) = \psi(x + 1)$ for any $x \in S \times \mathbb{R}$.

In such a way, the set S is a generator of a domain with a very simple dynamics.

Here are the main results of the article.

Theorem 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving self homeomorphism of the plane with a finite fixed points set $Fix(f) = P$. Suppose that for some metrics ρ in $\mathbb{R}^2 \backslash P$ there are no chain recurrent points: $CR_\rho(f|_{\mathbb{R}^2\backslash P}) = \emptyset$. Then there is a proper 1-dimensional foliation of $\mathbb{R}^2 \backslash P$ such that any leaf C is free and together with $f(C)$ is bounding a free domain homeomorphic to $\mathbb{R}^2$.*

We refer to the foliations described above as "Brouwer foliations". The leafs of a Brouwer foliation are called "Brouwer leafs".

Corollary 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving self homeomorphism of the plane with a finite fixed points set P. Suppose that for any subset of $A \subset P$ the sum of indices of all points in A does not equal 1. Then $\mathbb{R}^2 \backslash P$ admits a foliation by proper lines such that any leaf l is free and is generating a translation domain.*

Corollary 2. *(Patrice Le Calvez [8]) Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be an orientation preserving fixed points free self homeomorphism of the plane. Then there is a foliation of* $\mathbb{R}^2$ *by proper lines such that any leaf* l *is free and is separating* $f(l)$ *from* $f^{-1}(l)$*.*

Theorem 2. *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be an orientation preserving self homeomorphism and* p_0 *be an isolated point of the fixed points set* $Fix(f)$*. Suppose that* $Index(p_0) \neq 1$*. Let* U *be an open neighborhood of* p_0 *not containing other fixed points of* f*. Then* $f|_U$ *may be extended to a homeomorphism of the plane* $\widetilde{f}$ *such that* $Fix(\widetilde{f}) = \{p_0\}$*, there is a foliation of* $R^2 \backslash \{p_0\}$ *by proper lines, any leaf* l *is free and is generating a translation domain for* $\widetilde{f}$*.*

Remark 1. *In the setting of Theorem 1, the leafs may be either topological lines, or topological circles. The second case is illustrated by the simple example of dilatation:* $f(x) = \lambda x$ *with* $\lambda > 0$*. Then we may take the foliation of circles centered at the origin.*

Theorem 1 may be considered as a generalization of a result of E.Slaminka [16]. He proves that if f is an orientation preserving self homeomorphism with finitely many fixed points, which is a *free map* in the sense of M.Brown then through any point $x \notin Fix(f)$ passes a line l as above. Of course, he doesn't say anything about the existence of a foliation of such lines, so Slaminka's result may be viewed as a local one. Another difference is that we deal with a larger class of maps f, since for example the dilatation $f(x) = \lambda x$ is *not* a free map, although $CR_\rho(f|_{\mathbb{R}^2 \backslash \{O\}}) = \emptyset$ with respect to any metrics and the assumptions of Theorem 1 are available.

Let us note that Theorem 1 may be given a more invariant form, not referring to any metrics, but we prefer here the metrical approach in order to apply the results of M.Hurley [17] on existence of Lyapunov functions.

Remark 2. *The constructed foliations are piecewise smooth, but it seems that with some more care they may be done smooth (class* C^∞*).*

3 Strategy of the proofs

Here we shall give a sketch of the proof of the main results. The full version of the article with complete proofs will be published elsewhere.

In the proof of Theorem 1 we first refer to the paper of M.Hurley [17] where he proves some general results about the existence of Lyapunov functions in noncompact spaces. These imply the existence of a global Lyapunov function in $\mathbb{R}^2 \backslash P$ i.e. a continuous function which is strictly increasing along orbits (see the definition above). This is in fact the idea of John Franks in his proof of Brouwer's Plane Translation Theorem [7]. Then, following Franks' method, we approximate it in the Whitney topology by a smooth Morse function, which is still a Lyapunov function. We may suppose that to each critical value corresponds only one critical point, i.e. that the Lyapunov function is a *regular* Morse function. Then its level sets define a "quasi-foliation" of $\mathbb{R}^2 \backslash P$ with all "leaves" being free - i.e. disjoint from their images.

Next, we modify this "quasi-foliation" by pushing out its singular points at infinity. In such a way we obtain a true proper foliation of $\mathbb{R}^2 \backslash P$ with all leafs being free. The leafs may be either topological lines, or topological circles. But this is not at all the desired foliation, since some lines l may not separate $f(l)$ from $f^{-1}(l)$. We name such lines "fake lines". They are not relevant to the dynamics of f. "Fake circles" are impossible.

The last, but most technical and delicate part of the proof, is to modify appropriately this foliation in order to eliminate the fake lines.

Theorem 2 follows more or less easily from Theorem 1. First we extend f from some open $U \ni p_0$ to a homeomorphism $\widetilde{f}$ of the plane, which is fixed point free in $\mathbb{R}^2\backslash\{p_0\}$. Now, it is possible to define a metrics ρ in $\mathbb{R}^2\backslash\{p_0\}$ such that $\rho(x, f(x) \geq 1$ for any x. Then the condition $Index(p_0) \neq 1$ together with Franks' Lemma imply that $CR_\rho(f|_{\mathbb{R}^2\backslash\{p_0\}}) = \emptyset$. Hence Theorem 1 defines some foliation $\mathcal{F}$ with Brouwer leafs. Then we notice that the condition $Index(p_0) \neq 1$ implies that the map $f|_{\mathbb{R}^2\backslash\{p_0\}}$ is *free* in the sense of Brown [16]. Now it is straightforward to show that each leaf l of the foliation $\mathcal{F}$ defines a translation domain.

Let us note that Corollary 1 follows from Theorem 1 in a similar way.

We proceed with some details from the proof of Theorem 1.

Let $f : X \to X$ be a homeomorphism of the metric space X.

Definition 2. *A continuous function $\varphi : X \to \mathbb{R}$ is called Lyapunov function, if*

$$\varphi(x) < \varphi(f(x)) \quad \text{for any } x \in X.$$

The condition $CR_\rho(f|_{\mathbb{R}^2\backslash P}) = \emptyset$ in Theorem 1 implies, as follows from a result of M.Hurley [17], that there is a Lyapunov function $\varphi : \mathbb{R}^2\backslash P \to \mathbb{R}$.

Then φ may be approximated in Whitney topology by a smooth regular Morse function (c.f. [18]). In such a way one finds a Morse function which is a Lyapunov function as well. We shall denote it for simplicity by φ again. The level sets of φ are free with respect to f and define a "quasi-foliation" of $\mathbb{R}^2\backslash P$, i.e. a foliation with singularities, which are of 2 types: sinks and saddles. Now, in both cases it is possible to modify the "quasi-foliation" in order to obtain a true foliation $\mathcal{F}$ with free leafs. Moreover, these leafs are free in some stronger sense, namely, for every such leaf l we have $l \cap f^n(l) = \emptyset$, for any $n \neq 0$.

Of course, we may have "fake leafs" which are not relevant to the dynamics of f.

Definition 3. *A leaf l of the foliation $\mathcal{F}$ is said to be "fake", if it is the boundary of a free open region of $\mathbb{R}^2\backslash P$. We shall denote this region by $A(l)$. So, $\partial A(l) = l$, $f(A(l)) \cap A(l) = \emptyset$.*

If the leaf l is not fake, we shall say that it is a "Brouwer leaf".

It is easy to see that if l is a fake leaf, then $A(l)$ is an union of fake leafs as well. A fake leaf may be only a "loop", i.e. a topological line with coinciding ends from the set $P \cup \{\infty\}$.

Furthermore, it is not difficult to see that a Brouwer leaf l is bounding together with $f(l)$ some free region. In such a way, all we have to do is to eliminate the fake leafs.

If l and l' are fake leafs, we write $l \succ l'$, if $A(l) \supset A(l')$. Then it may be shown that if l_α is a collection of incomparable fake leafs, they form a *discrete* family - every compact set of $\mathbb{R}^2\backslash P$ is intersecting only a finite number of them. This allows us to define a disjoint family of "critical fake zones", that is - maximal open connected and free regions consisting of fake leafs. This family is discrete - every compact set is intersecting only a finite number of critical zones.

It may be seen that if W is a critical zone, then ∂W cannot touch > 2 elements of $P \cup \{\infty\}$. In case it touches exactly 2 elements, say a and b, it is easy to show that there are 2 leafs from ∂W connecting a and b and bounding a free region $W' \supset W$. Then we may fill up W' with a family of lines connecting a and b. Recall that each such line is a "Brouwer line". In such a way we modify the foliation $\mathcal{F}$ in some free region W' and cover W by Brouwer leafs.

The crucial case is the elimination of a critical fake zone W which touches only one element of $P \cup \{\infty\}$, say $\{\infty\}$. There are 2 possibilities here:

1) ∂W is the union of 2 Brouwer leafs. This is the easy case - one foliates W by lines which are separating the two components of ∂W.

2) ∂W is a single fake leaf l_0. This is the most delicate case. Then we notice that by the maximality property of W, there is an invariant open and connected $W' \supset W$ such that

$W'\backslash\overline{W}$ is filled up by Brouwer leafs. Then we modify the foliation in W' making the Brouwer leafs from $W'\backslash\overline{W}$ "penetrate" in W. In such a way, we cover W by Brouwer leafs. Of course, one should take considerable care in order to construct *free* leafs. Furthermore, we show that each such leaf, together with its image, is bounding a free domain homeomorphic to $\mathbb{R}^2$.

Now, representing $\mathbb{R}^2\backslash P$ as an union of some increasing sequence of compact sets K_n and noticing that each one of them intersects only a finite number of critical fake zones, we proceed by induction the modification of the foliation in the critical zones, finally obtaining the desired Brouwer foliation.

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БРАУЕРОВИ СЛОЕНИЯ В НЕРЕКУРЕНТНИ ОБЛАСТИ

С. Стефанов

Ключови думи: *теорема на Брауер за транслацията, слоение, неподвижна точка, индекс*

РЕЗЮМЕ

Запазващите ориентацията хомеоморфизми $f : \mathbb{R}^2 \to \mathbb{R}^2$ с крайно множество от неподвижни точки P са разгледани в настоящата статия. Доказано е, че ако рестрикцията $f|_{\mathbb{R}^2 \setminus P}$ няма рекурентни точки относно някаква метрика, то $\mathbb{R}^2 \setminus P$ допуска собствено 1-мерно слоение $\mathcal{F}$ такова че всеки негов слой S е свободен (т.е. не пресича образа си) и заедно с $f(S)$ загражда свободна област (Теорема 1). Освен това, ако f е свободно изображение, то всеки слой на $\mathcal{F}$ дефенера *транслационна област* (Забележка 2). Нерекурентна и свободна динамика имаме, например, ако сумата от индексите на всяко подмножество на P е различна от 1 (Следствие 1).

По-нататък, показано е, че ако p_0 е единствената неподвижна точка с топологичен индекс $\neq 1$ на един запазващ ориентацията локален хомеоморфизъм $f : U \to \mathbb{R}^2$, то съществува слоение на U с хубави динамични свойства. Именно, за всяко продължение $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ на f с единствена неподвижна точка p_0, съществува слоение на $\mathbb{R}^2$ от собствени топологични прави, такова че всеки слой е дизюнктен от своя образ и поражда транслационна област.