ГОДИШНИК НА УНИВЕРСИТЕТА ПО АРХИТЕКТУРА, СТРОИТЕЛСТВО И ГЕОДЕЗИЯ
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ABOUT WEIGHT PROPERTIES OF SPACES[1]

M. Choban[2], **E. Mihaylova**[3]

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ABSTRACT

In this paper, the new notion of an $\mathcal{A}$-space is introduced. A positive answer to a question of Arhangel'skii and Bella is given.

1 Introduction

In this paper, a space is meant to be a regular topological space. The terminology from [11] is used. The weight of a space X is denoted by $w(X)$. A family $\mathcal{S}$ of subsets of a space X is a network for X if for any point $x \in X$ and any neighborhood U of the point x there exists an element $P \in \mathcal{S}$ such that $x \in P \subseteq U$ (see [1]). The network weight of a space X is the smallest cardinal number of the form $|\mathcal{S}|$, where $\mathcal{S}$ is a network for X and is denoted by $nw(X)$.

A *remainder* of a Tychonoff space X is the subspace $Y \setminus X$ of a Tychonoff extension Y of X. The space Y is an extension of X if X is a dense subspace of Y.

This paper considers what kind of remainders a metric space can have. Distinct properties of remainders were studied in [3–7, 12].

By $\mathcal{A}$, the class of spaces with the following property is denoted:

(α) For any closed subspace Y of a space X we have $nw(Y) = w(Y)$.

The main result of the present paper is the following theorem.

Theorem 1. *Let $X \in \mathcal{A}$ and Y be a dense metrizable subspace of the space X. Then $X \setminus Y \in \mathcal{A}$ and in particular $nw(X \setminus Y) = w(X \setminus Y)$.*

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[2]M. Choban, Department of Mathematics, Tiraspol State University, Iablochikin 5, MD 2069, Kishinev, Republic of Moldova,e-mail: mmchoban@gmail.com

[3]E. Mihaylova, Department of Descriptive geometry and Engineering graphics, University of Architecture, Civil Engineering and Geodesy, 1046 Sofia, Bulgaria, e-mail: katiamih_fgs@uacg.bg

In particular, the following assertion contains a positive answer to Question 2.15 from [6].

Corollary 2. *If X is a compactification of a metrizable space Y, then $X \setminus Y \in \mathcal{A}$ and in particular $nw(X \setminus Y) = w(X \setminus Y)$.*

2 On the class $\mathcal{A}$

In this section, some notions are listed that were introduced in [2, 9, 10, 13].

A sequence $\{U_n : n \in \omega\}$ of open subsets of a space X is called a *compact sequence* if it satisfies the following conditions:

(S1) $\emptyset \neq U_{n+1} \subseteq cl_X U_{n+1} \subseteq U_n$ for any $n \in \omega$;

(S2) Every sequence $\{x_n \in U_n : n \in \omega\}$ has an accumulation point in X;

(S3) $\cap\{U_n : n \in \omega\}$ is a compact set.

Let X be a space , $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \omega\}$ be a sequence of families of open subsets of X, and let $\pi = \{\pi_n : A_{n+1} \to A_n : n \in \omega\}$ be a sequence of mappings. A sequence $\alpha = \{\alpha_n : n \in \omega\}$ is called a *c-sequence* if $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for every n. Let $H(\alpha)$ $= \cap\{U_{\alpha_n}; n \in \omega\}$. A c-sequence α is called marked if $H(\alpha)$ is a non-empty set. Consider the following conditions:

(SC1) $X = \cup\{U_\beta : \beta \in A_n\}$ for each $n \in \omega$;

(SC2) $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = \cup\{cl_X U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha$ for all $\alpha \in A_n$ and $n \in \omega$;

(SC3)) For any marked c-sequence $\alpha = \{\alpha_n \in A_n : n \in \omega\}$, the sequence $\{U_{\alpha_n}; n \in \omega\}$ is compact;

(SC4) any c-sequence is marked.

Sequences γ and π are called an A-sieve if they possess Properties (SC1) and (SC2). A space X is called *sieve-complete* if there exists an A-sieve with Properties (SC3) and (SC4). Any Čech-complete space is sieve-complete [9, 10, 13].

A space X is called *an A(k)-space* or a monotonically p-space if there exists an A-sieve with Property (SC3) [9, 10, 13].

A space X is called a *p-space* if X is a Tychonoff space and in some compactification bX of X there exists a sequence of open families $\gamma = \{\gamma_n : n \in \omega\}$ such that $x \in \bigcap\{\gamma_n(x) : n \in \mathbb{N}\} \subseteq X$ for each point $x \in X$, where $\gamma_n(x)$, $n \in \mathbb{N}$, is the star of the point x (see [2]). Every p-space is an $A(k)$-space [9, 10, 13].

By virtue of some results from [2, 10], one obtains:

Theorem 3 (see [2]). *If X is a p-space, then $X \in \mathcal{A}$.*

Theorem 4 (see [10]). *If X is an $A(k)$-space, then $X \in \mathcal{A}$.*

Proposition 5. *Assume that for each point $x \in X$ there exist an open set $U(x)$ and a subspace $Y(x)$ such that $Y(x) \in \mathcal{A}$ and $x \in U(x) \subseteq Y(x)$. Then $X \in \mathcal{A}$.*

Proof. Let $\mathcal{S}$ be a network of the space X. Then there exists a subset $L \subseteq X$ such that $|L| \leq |\mathcal{S}|$ and $\cup\{U(x) : x \in L\} = X$. Then $w(Y(x)) = nw(Y(x)) \leq |\mathcal{S}|$ and in $U(x)$ there exists an open base $\mathcal{B}(x)$ of the cardinality $|\mathcal{B}(x)| \leq |\mathcal{S}|$. Then $\mathcal{B} = \cup\{\mathcal{B}(x) : x \in L\}$ is a base for X and $|\mathcal{B}| \leq |\mathcal{S}|$. Hence $w(X) \leq nw(X)$. The proof is complete.

Corollary 6. *If Y is an open subspace of a space X and $X \in \mathcal{A}$, then $Y \in \mathcal{A}$.*

3 The main results

Theorem 7. *Let $X \in \mathcal{A}$, Y be a dense subspace of the space X, and in X there exists* *an open point-countable family $\mathcal{B}$ which forms a base for X in the points of the set Y. Then $X \setminus Y \in \mathcal{A}$.*

Proof. Fix a closed subspace Z of the space $X \setminus Y$. Let $\mathcal{S}$ be a network of Z and $\tau = nw(Z) = |\mathcal{S}|$. For each $L \in \mathcal{S}$ fix a point $z_L \in L$. Let $D = \{z_L : L \in \mathcal{S}\}$. Put $\mathcal{B}_1 = \{U \in \mathcal{B} : U \cap D \neq \emptyset\}$, $X_1 = cl_X Z$ and $Y_1 = Y \cap X_1$. Obviously $|\mathcal{B}_1| \leq \tau$ and $\mathcal{B}_1$ is a base for X_1 at the points of the set Y_1. Hence $\tau = nw(Z) \leq nw(X_1) \leq \tau + |\mathcal{B}_1| = \tau$ and $w(Z) \leq w(X_1) = nw(X_1) \leq \tau = nw(Z)$. Therefore $X \setminus Y \in \mathcal{A}$. The proof is complete.

Theorem 1 follows from the following assertion:

Corollary 8. *Let $X \in \mathcal{A}$, Y be a dense subspace of the space X and Y be a space with a σ-disjoint base. Then $X \setminus Y \in \mathcal{A}$.*

Theorem 9. *Let $f : X \longrightarrow Y$ be a perfect mapping of a completely regular space X onto a space Y. If $Y \in \mathcal{A}$, then $X \in \mathcal{A}$.*

Proof. It is sufficient to prove that $w(X) \leq nw(X)$. Let $\mathcal{S}$ be a network of X and $\tau = nw(X) = |\mathcal{S}|$. Since $nw(Y) \leq nw(X)$, one obtains $w(Y) \leq \tau$. Let βX be the Stone-Čech compactification of the space X. Denote by $\{(H_\alpha, F_\alpha) : \alpha \in M\}$ the family of all ordered pairs (H, F), where $H \in \mathcal{S}$ and $F \in \mathcal{S}$, such that $cl_{\beta X} H \cap cl_{\beta X} F = \emptyset$. For each $\alpha \in M$, a continuous function $g_\alpha : X \longrightarrow I = [0,1]$ is fixed such that $H_\alpha \subseteq g_\alpha^{-1}(0)$ and $F_\alpha \subseteq g_\alpha^{-1}(1)$. Let $g : X \longrightarrow Z \subseteq I^M$ be defined, where $Z = g(X)$ and $g(x) = (g_\alpha(x) : \alpha \in M)$ for each $x \in X$. The mappings g is continuous and one-to-one. By construction $w(Z) \leq |M| = \tau$. Now the mapping $\varphi : X \longrightarrow Y \times Z$ is considered, where $\varphi(x) = (f(x), g(x))$ for each $x \in X$. By virtue of Theorem 3.7.9 from [11], the mapping φ is a one-to-one perfect mapping of the space X onto the subspace $\varphi(X)$ of $Y \times Z$. Thus $w(X) \leq w(Y) + w(Z) \leq \tau$. The proof is complete.

References

[1] A. V. Arhangel'skii. *An addition theorem for the weight of sets lying in bicompacta*, Doklady Acad. Nauk SSSR **126** (1959), 239-241.

[2] A. V. Arhangel'skii, *A class of spaces which contains all metric and all locally compact spaces*, Matem. Sb. 67 (1965), 55-88 (English translation:t Amer. Math. Soc. Transl. 92 (1970), 1-39).

[3] A. V. Arhangel'skii, *Remainders in compactifications and generalized metrizability properties*, Topology and Appl. **150** (2005), 79-90.

[4] A.V. Arhangel'skii, *Remainders in compactifications and generalized metrizability properties*, Topology and Appl. **150** (2005), 79-90.

[5] A.V. Arhangel'skii, *More on remainders close to metrizable spaces*, Topology and Appl. **154** (2007), 1084-1088.

[6] A. V. Arhangel'skii and A. Bella. *Cardinal invariants in remainders and variations of tightness*, Proceed. Amer. Math. Soc. **119:3**, (1993), 947-954.

[7] A. V. Arhangel'skii and M. M. Choban, *Remainders of rectifiable spaces*, Topology and Applications **157** 4, (2010), 789-799.

[8] A. V. Arhangelskii and M. G. Tkachenko, *Topological groups and related structures*, Atlantis Press. Amsterdam-Paris, 2008.

[9] J. Chaber, M. M.Čoban and K. Nagami, *On monotonic generalizations of Moore spaces, Čech complete spaces and p-spaces,* Fund. Math. **84** (1974), 107-119.

[10] M. Choban, *The open mappings and spaces*, Suplimente ai Rendicanti del Circolo Matematico di Palermo. Serie II, numero **29** (1992), 51-104.

[11] R. Engelking, *General Topology*, PWN. Warszawa, 1977.

[12] M. Henriksen and J. R. Isbel, *Some properties of compactifications,* Duke Math. Jour. **25** (1958), 83-106.

[13] H. H. Wicke, *Open continuous images of certain kinds of M-spaces and completeness of mappings and spaces,* General Topology and Appl. **1** (1971), 85-100.

ОТНОСНО ТЕГЛОВИТЕ СВОЙСТВА НА ПРОСТРАНСТВАТА[4]

М. Чобан[5], Е. Михайлова[6]

Ключови думи: *топология, компактни пространства, топологично тегло*

РЕЗЮМЕ

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[5]М. Чобан, Тираспoлски Университет, ул. Яблочкин 5, Кишинев 2069, Молдова, ел. адрес: mmchoban@gmail.com

[6]Е. Михайлова, Кат. Дескриптивна геометрия и ИСГ, УАСГ, 1046 София, ел. адрес: katiamih_fgs@uacg.bg