Analytical Least Squares Design of 2-D Fan Type FIR Filter

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Abstract : In this paper, an analytical LS technique for design of 2-D FIR filters is presented. Our attention is focused to the design of fan type 2-D filter. The coefficients of the filter are obtained by solving a system of linear equations. Also, closed-form expressions are derived for the elements of the matrices which appear in LS approach. This analytical solution enables fast calculation and simplicity of the overall filter design.

1. Introduction

Recently, the Least Squares (LS) method for the 2-D FIR filters design has obtained wide attention, since it is computationally more efficient than other minimax techniques. Pei and Shyu extended the 1-D eigenfilter approach [1] for designing 2-D quadrantally-symmetric FIR filters [2]. A similar idea is independently developed by Nashashibi and Charalambous [3]. Thus, the filter coefficients are found by computing the eigenvector corresponding to the smallest eigenvalue of a real, symmetric and positive-definite matrix. A novel WLS method is given in [4] which retains the frequency samples and impulse filter coefficients in 2-D form. The application of WLS algorithm for zero-phase 2-D FIR filters is presented by Gislason, et.al. [5]. A very useful procedure for 2-D linear-phase FIR filters is proposed by the authors Kidambi and Ramachandran [6]. There, a more meaningful formulation than the eigenfilter method has been established without using a reference frequency. The coefficients of the filter are obtained by solving a system of linear equations.

An analytical technique for the LS design of 2-D zero-phase FIR filters that have a general-shaped centro-symmetric frequency response is proposed in [7]. Other alternative investigations concerning the WLS approach for 2-D filters have been presented in [8-11].

In this paper, we give an analytical weighted LS technique for design of 2-D FIR filters, and focus our attention on obtaining closed-form expressions for the fan type 2-D filter. This analytical solution enables fast calculation of the filter coefficients and decreases the computational complexity as a whole. A similar approach can be developed for some other specific types of 2-D filters: rectangular, half-band, etc.

2. Two-Dimensional Quadrantally-Symmetric FIR Filters

The frequency response of a 2-D nonrecursive filter with N_1 by N_2 taps is given by [10] :

$$H(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} h(n_1, n_2) \cdot e^{-jn_1\omega_1} \cdot e^{-jn_2\omega_2}, \tag{1}$$

where $h(n_1, n_2)$ is the impulse response of the filter. For a 2-D quadrantally-symmetric filter with N_1 and N_2 odd, the following relationship holds :

$$h\left(\frac{N_1-1}{2}-k_1, \frac{N_2-1}{2}-k_2\right) = h\left(\frac{N_1-1}{2}-k_1, \frac{N_2-1}{2}+k_2\right)$$
$$= h\left(\frac{N_1-1}{2}+k_1, \frac{N_2-1}{2}-k_2\right) = h\left(\frac{N_1-1}{2}+k_1, \frac{N_2-1}{2}+k_2\right) \tag{2}$$

for $1 \le k_1 \le (N_1 - 1)/2$, $1 \le k_2 \le (N_2 - 1)/2$. Then (1) can be written in the form [12] :

$$H(e^{j\omega_1}, e^{j\omega_2}) = M(\omega_1, \omega_2).e^{-j\frac{N_1-1}{2}\omega_1}.e^{-j\frac{N_2-1}{2}\omega_2}, \tag{3}$$

where $M(\omega_1, \omega_2)$ is the amplitude response given by :

$$M(\omega_1, \omega_2) = \sum_{n_1=0}^{\frac{N_1-1}{2}} \sum_{n_2=0}^{\frac{N_2-1}{2}} a(n_1, n_2).\cos n_1\omega_1.\cos n_2\omega_2 . \tag{4}$$

The coefficients $a(n_1, n_2)$ are related to the $h(n_1, n_2)$ by:

$$a(0,0) = h\left(\frac{N_1-1}{2}, \frac{N_2-1}{2}\right);$$
$$a(0,n_2) = 2h\left(\frac{N_1-1}{2}, \frac{N_2-1}{2}-n_2\right),\ n_2 = 1,\dots,(N_2-1)/2$$
$$a(n_1,0) = 2h\left(\frac{N_1-1}{2}-n_1, \frac{N_2-1}{2}\right),\ n_1 = 1,\dots,(N_1-1)/2 \tag{5}$$
$$a(n_1,n_2) = 4h\left(\frac{N_1-1}{2}-n_1, \frac{N_2-1}{2}-n_2\right),\ n_{1,2} = 1,\dots,(N_{1,2}-1)/2 .$$

3. A Closed-Form LS Solution to the 2-D Fan Type Filter Design Problem

The mean-square error between the desired amplitude response $D(\omega_1, \omega_2)$ and the actual one $M(\omega_1, \omega_2)$ can be formulated as :

$$E = \alpha.\iint_p \left[D(\omega_1, \omega_2) - M(\omega_1, \omega_2)\right]^2 d\omega_1 d\omega_2 +$$
$$+ \beta.\iint_S M^2(\omega_1, \omega_2)\, d\omega_1 d\omega_2 = \alpha E_p + \beta E_S , \tag{6}$$

where p is the passband and s is the stopband in the (ω_1, ω_2) plane. The quantities α and β are weighted constants in the passband and stopband, respectively

By minimizing the error function E we can obtain a system of linear equations [6] :

$$(\alpha.\mathbf{Q} + \beta.\mathbf{R}).\mathbf{a} = \alpha.\mathbf{d} , \tag{7}$$

where : $\mathbf{Q} = \iint\limits_p \mathbf{c}(\omega_1,\omega_2).\mathbf{c}^{\mathbf{T}}(\omega_1,\omega_2)d\omega_1 d\omega_2$,

$$\mathbf{R} = \iint\limits_s \mathbf{c}(\omega_1,\omega_2).\mathbf{c}^{\mathbf{T}}(\omega_1,\omega_2)d\omega_1 d\omega_2 ,$$

$$\mathbf{d} = \iint\limits_p D(\omega_1,\omega_2).\mathbf{c}(\omega_1,\omega_2)d\omega_1 d\omega_2 .$$

The column vectors $\mathbf{a}$ and $\mathbf{c}(\omega_1,\omega_2)$ from eq.(7) are defined as follows :

$$\mathbf{a} = \Big[a(0,0)\ ,\ a(0,1), \ldots, a\Big(0,\frac{N_2-1}{2}\Big)\ \mathrm{M}\, a(1,0), \ldots, a\Big(1,\frac{N_2-1}{2}\Big)\ \mathrm{M}..$$

$$\ldots \mathrm{M}\, a\Big(\frac{N_1-1}{2},0\Big), a\Big(\frac{N_1-1}{2},1\Big), \ldots, a\Big(\frac{N_1-1}{2},\frac{N_2-1}{2}\Big)\Big]^T \qquad (8a)$$

and

$$\mathbf{c}(\omega_1,\omega_2) = \Big[1,\ \cos\omega_2, \ldots,\ \cos\frac{N_2-1}{2}\omega_2\ \mathrm{M}\cos\omega_1,\ \cos\omega_1.\cos\omega_2, \ldots$$

$$\ldots,\ \cos\omega_1.\cos\frac{N_2-1}{2}\omega_2\ \mathrm{M}..\ \mathrm{M}\cos\frac{N_1-1}{2}\omega_1,\ \cos\frac{N_1-1}{2}.\cos\omega_2,\ \ldots$$

$$\ldots,\ \cos\frac{N_1-1}{2}\omega_1.\cos\frac{N_2-1}{2}\omega_2\Big]^T . \qquad (8b)$$

As can be seen $\mathbf{Q}$ and $\mathbf{R}$ are positive-definite, real and symmetric matrices. If we solve the upper defined system (7), the amplitude response of the filter can be expressed as :

$$M(\omega_1,\omega_2) = \mathbf{a}^{\mathbf{T}}.\mathbf{c}(\omega_1,\omega_2) . \qquad (9)$$

The aim of this paper is to consider the application of the LS criterion to the case of 2-D fan type filter. Also, closed-form expressions will be derived for the elements of the matrices which appear in system (7). It will be shown that these expressions depend on the desired fan filter specifications :

$$D(\omega_1,\omega_2) = \begin{cases} 1 & p:\ 0 \le \omega_1 \le \pi\ ,\ \omega_1 \le \omega_2 \le \pi \\ 0 & s\ :\ \omega_a \le \omega_1 \le \pi\ ,\ 0 \le \omega_2 \le \pi - \omega_a \end{cases} , \qquad (10)$$

where ω_a is the stopband cutoff frequency.

4. Evaluation of the elements of matrices

At first, let us consider the vector $\mathbf{d}$ from (7). Since $D(\omega_1,\omega_2) = 1$ in the passband of fan filter, we can rewrite $\mathbf{d}$ as :

$$\mathbf{d} = \iint\limits_p \mathbf{c}(\omega_1,\omega_2)d\omega_1 d\omega_2 . \qquad (11)$$

Here, $\mathbf{c}(\omega_1,\omega_2)$ has t number of elements, where t=(N_1+1)(N_2+1)/4. Applying (8b) and the common relation :

$$\iint\limits_p \mathbf{c}(\omega_1,\omega_2)d\omega_1 d\omega_2 = \int_0^{\pi}\Big[\int_{\omega_1}^{\pi} \cos a\omega_1 \cdot \cos c\omega_2\, d\omega_2\Big]d\omega_1$$

for a = 0,1,..., (N_1-1)/2 , c = 0,1,..., (N_2-1)/2 ,

we can obtain the final expressions for the elements of $\mathbf{d}$ given in Table 1.

$\{d_i\}$, i = 1, 2,..., t	Conditions
$\frac{1}{2c(a+c)}\cos(a+c).\pi + \frac{1}{2c(c-a)}\cos(c-a).\pi - \frac{1}{c^2-a^2}$	$a \ne c$, $c \ne 0$
$\frac{1}{a^2}(1-\cos a\pi)$	$c = 0$, $a \ne 0$
0	$a = c \ne 0$
$\pi^2/2$	$a = c = 0$

Table 1

By analogy, we evaluate the integral :

$$\mathbf{Q} = \iint\limits_p \mathbf{c}(\omega_1,\omega_2).\mathbf{c}^{\mathbf{T}}(\omega_1,\omega_2)d\omega_1 d\omega_2 =$$

$$= \int_0^{\pi}\Big[\int_{\omega_1}^{\pi} \cos a\omega_1 \cdot \cos b\omega_1 \cdot \cos c\omega_2 \cdot \cos d\omega_2\, d\omega_2\Big]d\omega_1 ,$$

where : a = 0,1,..., (N_1-1)/2 , b =0,1,..., (N_1-1)/2 ,
c = 0,1,..., (N_2-1)/2 , d =0,1,..., (N_2-1)/2 .

As a result, the elements of the matrix $\mathbf{Q} \equiv [Q_{ij}]$ with dimension t x t can be derived as :

$$[Q_{ij}] = A + B + C , \qquad (12)$$

where :

$$A = \frac{1}{8.z(x+z)}\cos\pi(x+z) + \frac{1}{8.z(z-x)}\cos\pi(z-x) + \frac{1}{8.z(z+y)}\cos\pi(z+y) + \frac{1}{8.z(z-y)}\cos\pi(z-y) ,$$

$$B = \frac{1}{8.r(x+r)}\cos\pi(x+r) + \frac{1}{8.r(r-x)}\cos\pi(r-x) + \frac{1}{8.r(r+y)}\cos\pi(r+y) + \frac{1}{8.r(r-y)}\cos\pi(r-y) ,$$

$$C = \frac{2x^2 - z^2 - r^2}{4(z^2-x^2)(r^2-x^2)} + \frac{2y^2 - z^2 - r^2}{4(z^2-y^2)(r^2-y^2)} ,$$

under the following conditions :
$z \pm x \ne 0$, $z \pm y \ne 0$, $r \pm x \ne 0$, $r \pm y \ne 0$, $z \ne 0$, $r \ne 0$. Also, in view of simplicity we use some extra substitutions :

$$x = a + b\ ,\ y = a - b\ ;$$
$$z = c + d\ ,\ r = c - d\ .$$

When $c = d \ne 0$, i.e. $r = 0$:

$$[Q_{ij}] = A + B' + C' , \qquad (13)$$

where :

$$B' = -\frac{1}{4x^2}\cos x\pi - \frac{1}{4y^2}\cos y\pi ,$$

$$C' = \frac{x^2 + y^2 - 2z^2}{4(z^2-x^2)(z^2-y^2)} + \frac{x^2+y^2}{4x^2y^2} .$$

By analogy for the case of $c = d = 0$, i.e. $z = 0$ the elements $\left[Q_{ij}\right]$ are :

$$\left[Q_{ij}\right] = 2.B' + \frac{x^2 + y^2}{2.x^2 y^2} \ . \tag{14}$$

In most cases when the relation (12) is undefined, we can prove that $\left[Q_{ij}\right] \equiv 0$ (see Table 2a). Some exceptions to this rule are available (Table 2b) for the elements located at the main diagonal of **Q**.

The elements of $\mathbf{R} \equiv \left[R_{ij}\right]$ can be expressed in terms of the frequency ω_a as follows :

$$\left[R_{ij}\right] = -\frac{1}{4} S.T \ , \tag{15}$$

where : $S = \frac{1}{z}\sin z(\pi - \omega_a) + \frac{1}{r}\sin r(\pi - \omega_a)$,

$T = \frac{1}{x}\sin x\omega_a + \frac{1}{y}\sin y\omega_a$,

for $z \neq 0$, $r \neq 0$, $x \neq 0$, and $y \neq 0$.

$\left[Q_{ij}\right]$	Conditions
0	$x = z$, $x \neq 0$, $z \neq 0$
	$r = x$, $r \neq 0$, $x \neq 0$
	$r = y$, $r \neq 0$, $y \neq 0$
	$x + r = 0$, $x \neq 0$, $r \neq 0$
	$y + z = 0$, $y \neq 0$, $z \neq 0$
	$y + r = 0$, $y \neq 0$, $r \neq 0$
	$a = c \neq 0$ $b = d \neq 0$, $a \neq b$
	$a = c = 0$ $b = d \neq 0$
	$a = c \neq 0$ $b = d = 0$
	$b = c \neq 0$ $a = d \neq 0$, $a \neq b$
	$b = c = 0$ $a = d \neq 0$
	$b = c \neq 0$ $a = d = 0$

Table 2a

$\left[Q_{ij}\right]$	Conditions
$\pi^2/4$	$a = b \neq 0$ $c = d = 0$
$\pi^2/4$	$a = b = 0$ $c = d \neq 0$
^{2}	$a = b \neq 0$ $c = d \neq 0$
^{2}	$a = b = c = d = 0$

Table 2b

When the condition $a{=}b \neq 0$ holds :

$$\left[R_{ij}\right] = -\frac{1}{4} S.T' \ , \tag{16}$$

where : $T' = \frac{1}{x}\sin x\omega_a - \pi + \omega_a$,

for $x \neq 0$, $z \neq 0$, $r \neq 0$.

By analogy for the case of $c{=}d \neq 0$:

$$\left[R_{ij}\right] = -\frac{1}{4} S'.T \ , \tag{17}$$

where : $S' = \frac{1}{z}\sin z(\pi - \omega_a) + \pi - \omega_a$,

for $z \neq 0$, $x \neq 0$, $y \neq 0$.

The rest of expressions for $\left[R_{ij}\right]$ can be summarized as follows :

$\left[R_{ij}\right] = \frac{T}{2}(\omega_a - \pi)$, for $c = d = 0$, $x \neq 0$, $y \neq 0$;

$\left[R_{ij}\right] = \frac{S}{2}(\pi - \omega_a)$, for $a = b = 0$, $z \neq 0$, $r \neq 0$;

$\left[R_{ij}\right] = (\pi - \omega_a)^2$, for $a = b = c = d = 0$.

5. Design examples

A software tool is created based on the above approach to check the correctness of the final relationships as well as to solve different design tasks. The input specifications include the orders of the 2-D filter (N_1 and N_2), stopband cutoff frequency (see eq.10) and the weighted constants α and β. Gaussian elimination procedure is applied to solve a set of linear equations. The MatLab graphical capabilities have been used to produce the magnitude response of the filter.

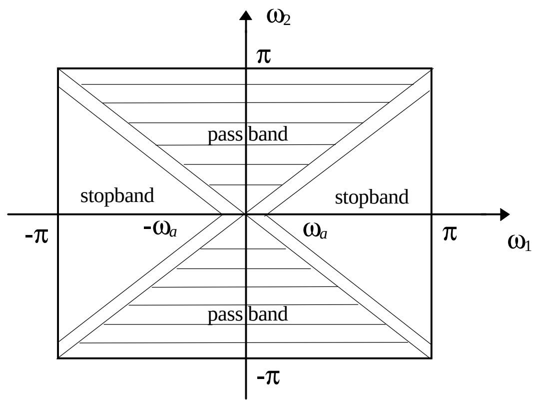

Fig.1 Specification chart of the designed fan filter

Fig.1 presents the specification chart of a 2-D fan filter in which ω_a is the stopband cutoff frequency. Setting $N_1{=}N_2{=}15$, $\omega_a = 0.16\pi$, $\alpha{=}\beta{=}1$, we get the fan filter magnitude response shown in Fig.2. The next figures - Fig.3 and Fig.4 give graphical results (the magnitude response and contour plot) for a 17 x 17 fan filter with $\omega_a = 0.3\pi$. The better design results are clearly visible with increasing the order of the filter. In addition, the overall computational time for the last example is about 20 seconds with PC-486/33 MHz.

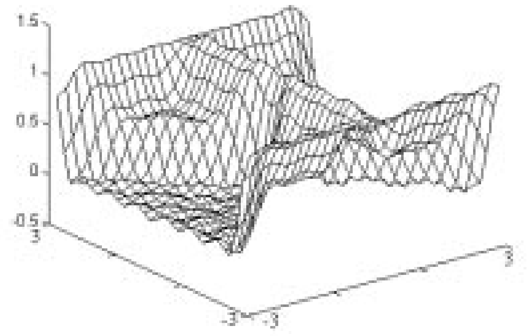

Fig.2 Magnitude response of the designed 15x15 fan filter with $\omega_a = 0.16\pi$

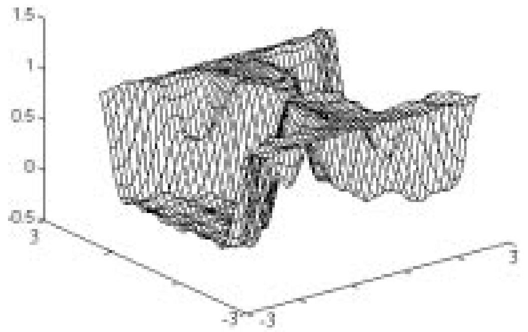

Fig.3 Magnitude response for 17x17 fan filter with $\omega_a = 0.3\pi$

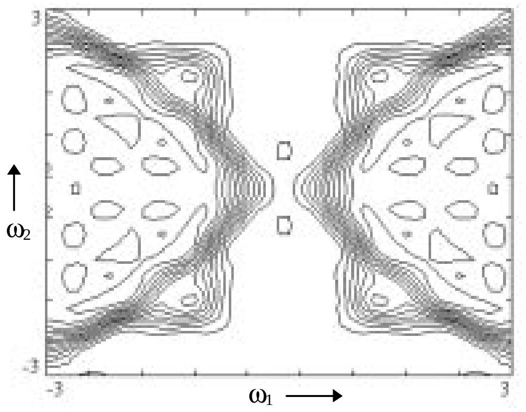

Fig.4 A contour plot for the filter from Fig.3 with a grid of 12 contour lines

Conclusion

This paper proposes some new expressions in analytical form which describe the LS design of 2-D fan filter. The results from the numerical examples show that we can save a lot of computational time comparing with the standard LS procedures which usually include numerical integration. Using the above approach, we are able to apply it towards another specific types of 2-D filters (rectangular, half-band, etc.).

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