Fullband Higher Order Differentiators Based on LS Method

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Abstract— **This paper deals with the Least Squares (LS) design of fullband higher order digital differentiators. The contribution extends a previous work on the problem presenting some new results (analytical, graphical and numerical) and conclusions. The design method for even and odd arbitrary *k*-th order differentiators based on the LS integral error criterion is considered. A few new closed-form relations for impulse and amplitude responses for LS fullband differentiators are proposed (with order *k*=1...4). The calculation of the impulse response of a differentiator is very simple and do not require solving a system of linear equations (as it is necessary in other LS techniques). Different numerical examples are shown using the Matlab source. Several amplitude responses and error curves are given. A comparison between fullband and non-fullband higher order differentiators with respect to the peak error in the passband is discussed.**

Keywords— **digital differentiators; least-squares method; fullband differentiators**

I. Introduction

Different methods have been proposed for designing higher order digital differentiators (DD) – eigenfilter method, minimax method, least-squares technique, etc. The design approaches are extensions of those used for first order differentiators. Rahenkamp and Vijaya Kumar [1] have modified the well-known McClellan-Parks program for the case of higher order digital differentiators. Unfortunately, this modified program often leads to very large deviation or fails to converge [2], especially for fullband higher order DD. An eigenfilter method has been proposed in [2,3]. A comparison to the McClellan-Parks algorithm shows that both methods are optimal in the sense of different minimum norms of the error function, but much better performance [2] is obtained with the eigenfilter design in the most of the frequency band. The eigen approach is based on the computation of an eigenvector of an appropriate real, symmetric, and positive-definite matrix. The elements of this matrix are usually evaluated by numerical integration (sometimes very time-consuming). In [3] it is shown, that an analytic solution is also possible, and, by that means, a design time of the eigenfilter approach is reduced greatly. Another analytic technique is given in [4], where a low relative error is obtained around the spot frequencies $\omega=\pi/$(any integer). A quadratic programming could be also used for the design of higher order DD [5] with a good accuracy at the mid-range frequencies.

The Least-squares (LS) design of higher order nonrecursive differentiators is introduced in [6,7]. The method leads to a lower mean-square error and is computationally more efficient than the eigenfilter method and the method based on the McClellan-Parks program. Also, it is not necessary to use a reference frequency as in the eigenfilter design. New analytical closed-form relations for higher order DD based on the LS design are proposed in [8].

This paper extends the previous work of the authors [8] and pays close attention to the design of fullband differentiators. Some new results (analytical, graphical and numerical) and useful conclusions are presented.

II. Fullband Differentiators

A. Design method

The magnitude response of an ideal k-th order differentiator is given by:

$$D(\omega)=(\omega/2\pi)^k\,,\ |\omega|\le\omega_p\,, \tag{1}$$

where ω_p denotes the passband edge frequency ($\omega_p=\pi$ for fullband case). The frequency response in this case is $H_I(e^{j\omega})=D(\omega).e^{jk\pi/2}$. Depending upon the values of k and ω_p, four types of differentiators are considered in the literature [1-9], namely, fullband odd-order, fullband even-order, non-fullband odd-order, and non-fullband even-order differentiators.

The aim of this paper is to consider the LS design of fullband differentiators (even and odd order) based on the technique introduced in [8].

By minimizing the mean-square error function:

$$Emse = \frac{1}{\pi} \int_0^{\omega_p} [D(\omega) - M(\omega)]^2 d\omega \qquad (2)$$

a required higher order differentiator in [8] is designed. $M(\omega)$ is the real magnitude response. It is well known that (2) leads to a system of linear equations. This system has been usually solved by computationally efficient methods, like the Cholesky decomposition [6,7].

Using the approach [8], we can avoid solving a system of linear equations for fullband higher order FIR differentiators. Odd-order fullband differentiators could be designed only when the filter length N is even. Similarly, even-order fullband differentiators are obtainable for odd N. Design requirements for these two types of fullband DD correspond to using Case IV and Case I linear-phase FIR filters with impulse response $h(n)$ (Table I).

B. Odd-order fullband differentiators

It was shown in [8] that for fullband odd-order DD ($\omega_p=\pi$, N even) the coefficients $b(n)$ can be expressed as:

$$b(n) = 2^{1-k} \sum_{l=1}^{(k+1)/2} \frac{(-1)^{l+n} r_1}{\pi^{2l}(n-1/2)^{2l}}, \quad 1 \le n \le N/2,$$

where $r_1=k(k-1)...(k-2l+2)$ for $1 \le l \le (k+1)/2$.

TABLE I

DESIGN REQUIREMENTS

	Fullband - odd k	Fullband - even k
FIR filter	Case IV	Case I
N	even	odd
$M(\omega)$	$\sum_{n=1}^{N/2} b(n)\sin(n-1/2)\omega$	$\sum_{n=0}^{(N-1)/2} b(n)\cos n\omega$
$H(e^{j\omega})$	$M(\omega)e^{j(\pi/2-\omega(N-1)/2)}$	$M(\omega)e^{-j\omega(N-1)/2}$
$h(n)$	$h(n) = -h(N-1-n)$ antisymmetric $h(N/2-n) = b(n)/2$ $1 \le n \le N/2$	$h(n) = h(N-1-n)$ symmetric $h((N-1)/2-n) = b(n)/2$ $1 \le n \le (N-1)/2$ $h((N-1)/2) = b(0)$

This allows a closed form relation for the magnitude response $M(\omega)$ to be written:

$$M(\omega) = 2^{1-k} \sum_{n=1}^{N/2} \left(\sum_{l=1}^{(k+1)/2} \frac{(-1)^{l+n} r_1}{\pi^{2l}(n-1/2)^{2l}} \right) \sin(n-1/2)\omega \qquad (3)$$

The expressions for the antisymmetric impulse response $h(n)$ and the designed amplitude response $M(\omega)$ for first and third order LS fullband differentiators are given in Table II. The parameter n_1 has been introduced as:

$$n_1 = \frac{(-1)^n}{\pi^2(n-1/2)^2}.$$

TABLE II

FIRST AND THIRD ORDER DIFFERENTIATORS

k	$h(N/2-n)$, $1 \le n \le N/2$	$M(\omega)$
1	$-n_1/2$	$-\sum_{n=1}^{N/2} n_1 \sin(n-1/2)\omega$
3	$\frac{3}{8} n_1 [2n_1(-1)^{-n} - 1]$	$\frac{3}{4}\sum_{n=1}^{N/2} n_1 [2n_1(-1)^{-n} - 1] \times$ $\times \sin(n-1/2)\omega$

C. Even-order fullband differentiators

For the even-order fullband case ($\omega_p=\pi$, N odd) the following relation has been obtained [8]:

$$b(n) = \begin{cases} 2^{1-k} \sum_{l=1}^{k/2} \frac{(-1)^{l+n+1} h}{(\pi n)^{2l}}, & 1 \le n \le (N-1)/2 \\ \frac{1}{2^k (k+1)}, & n = 0 \end{cases}$$

where $h=k(k-1)...(k-2l+2)$ for $1 \le l \le k/2$. By that means we can rewrite the magnitude response as:

$$M(\omega) = \frac{1}{2^k(k+1)} + 2^{1-k} \times \times \sum_{n=1}^{(N-1)/2} \left(\sum_{l=1}^{k/2} \frac{(-1)^{l+n+1} h}{(\pi n)^{2l}} \right) \cos n\omega . \qquad (4)$$

TABLE III

SECOND AND FOURTH ORDER DIFFERENTIATORS

k	$h((N-1)/2-n)$, $0 \le n \le (N-1)/2$	$M(\omega)$
2	$1/12$, $n=0$ $n_2/2$, $1 \le n \le (N-1)/2$	$1/12 + \sum_{n=1}^{(N-1)/2} n_2 \cos n\omega$
4	$1/80$, $n=0$ $\frac{n_2}{4}[1-6(-1)^{-n}]$, $1 \le n \le (N-1)/2$	$1/80 + \sum_{n=1}^{(N-1)/2} n_2 \times$ $\times [0.5 - 3(-1)^{-n}] \cos n\omega$

For second and fourth order differentiators, for example, the impulse and amplitude responses are given in Table III with a new parameter $n_2 = (-1)^n / (\pi n)^2$ introduced.

III. EXAMPLES

An ideal and designed amplitude responses for a 6th order fullband DD with N=41 are shown in Fig.1. Details for the frequencies $\omega \in [0, 2\pi/3]$ are given in the upper left corner of the figure. The coefficients of the differentiator are given in Table IV. Due to the symmetry of the coefficients for an even-order fullband DD, i.e. $h(n)=h(N-1-n)$, we only list $h(n)$ for n = 0...20. The peak error in the passband Epeak = max $|D(\omega)-M(\omega)|$ for this example has a value Epeak=9.2505e-4.

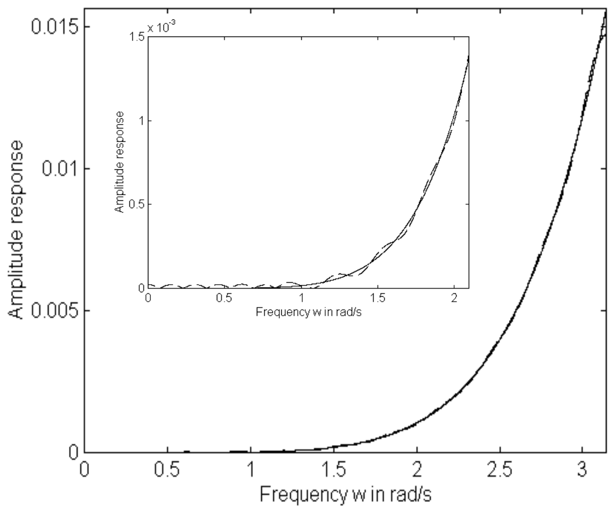

Fig.1. An ideal (solid line) and designed (dash dot line) amplitude responses.

TABLE IV
FULLBAND DIFFERENTIATOR COEFFICIENTS

n	$h(n)$	n	$h(n)$
0	2.362703077751e-5	11	-1.14358099135e-4
1	-2.616518064757e-5	12	1.437649477565e-4
2	2.913445314287e-5	13	-1.859368110007e-4
3	-3.263805023044e-5	14	2.492556547267e-4
4	3.681191102462e-5	15	-3.499054081554e-4
5	-4.183796560595e-5	16	5.213453995000e-4
6	4.796407059158e-5	17	-8.338423927449e-4
7	-5.553475116851e-5	18	0.00135451133013
8	6.503995587669e-5	19	-0.00195196081352
9	-7.719487531201e-5	20	0.00223214285714
10	9.307543980851e-5		

The plots of $20\log_{10}$Epeak against the order k for various values of N are shown in Fig.2 (only even-order differentiators are examined). It could be seen that for any N, the plots in the figure are approximately linear. This is also true for odd-order fullband DD.

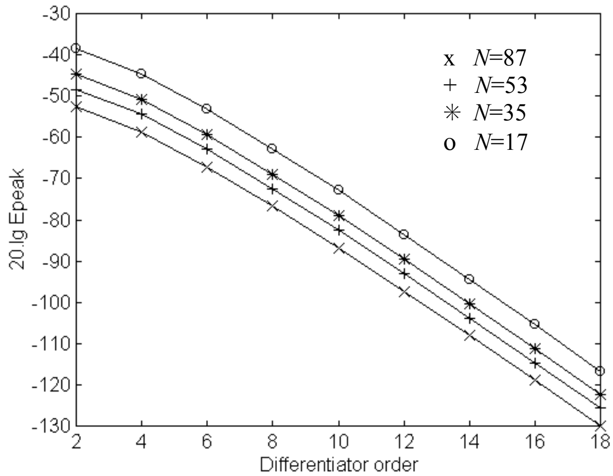

Fig.2. Variation of $20\log_{10}$Epeak with differentiator order for four different values of filter length N.

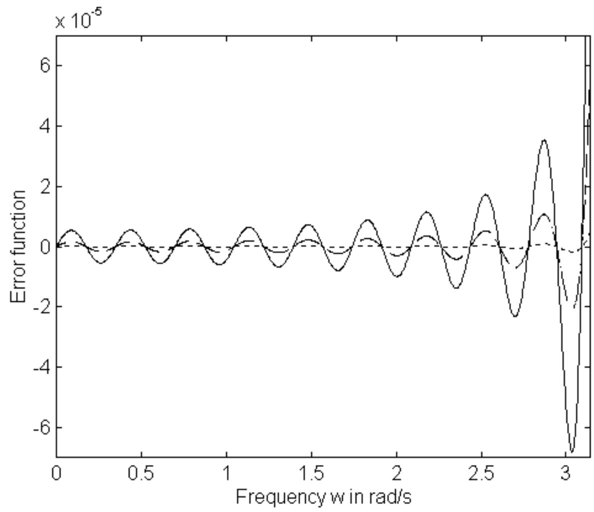

Fig.3. The error curves for fullband differentiators (N=36) with three different orders k (k=9 – solid line, k=11 – dashed line, k=15 – dotted line).

The variation of the approximation error $E(\omega)=D(\omega)-M(\omega)$ with respect to ω is given in Fig.3. Three different odd-order differentiators are examined – with orders k=9, 11, and 15. The peak errors in the passband are Epeak=1.9670e-4, 5.9891e-5, and 5.0569e-6, respectively. Obviously, the error is higher in the narrowband region near the passband edge frequency $\omega_p=\pi$. Also, an improvement of the error results is possible when N and/or k increase (see Fig.2 and Fig.3). The last two statements are valid for non-fullband differentiators, as well.

The results for Epeak (in dB) obtained from different numerical examples are presented in Table V. Differentiators (fullband and non-fullband) with N=5 and N=155 are examined. We achieve very good results for the accuracy of the method, especially for higher N and k (for example Epeak=-183.0591dB when N=155 and k=26). Four 5th order non-fullband differentiators are designed, too (see Examples 5÷8, Table V). We can see that when ω_p is close to π (e.g. $\omega_p=0.99\pi$) the fullband 4th order differentiator is more accurate than the non-fullband 5th order one. For lower ω_p and constant k the error for non-fullband DD decreases (clearly visible for higher values of N).

From the computational point of view, our method is more efficient than other alternative techniques which require solving a system of linear equations. Furthermore, a comparison given in [8] proves that the proposed method is more accurate than the minimax approach [1].

TABLE V
NUMERICAL EXAMPLES

Examples		Epeak, dB	
		N=5	N=155
1.	k=4, fullband	-34.2453	-63.6930
2.	k=12, fullband	-76.0220	-102.3205
3.	k=18, fullband	-110.8877	-134.9302
4.	k=26, fullband	-158.4891	-183.0591
5.	k=5, $\omega_p=0.99\pi$ non-fullband	-30.6825	-48.7549
6.	k=5, $\omega_p=0.98\pi$ non-fullband	-31.2808	-69.7545
7.	k=5, $\omega_p=0.97\pi$ non-fullband	-31.8971	-90.9873
8.	k=5, $\omega_p=0.96\pi$ non-fullband	-32.5303	-112.3288

IV. CONCLUSIONS

In this paper, an analytical design of fullband higher order FIR differentiators has been considered. The filter coefficients are found in a non-iterative simple manner using the exact mathematical formulas. The results demonstrate the effectiveness and accuracy of the proposed approach. Designed higher order differentiators can be used in different applications – in biological signal processing and calculation of geometric moments.

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