February 2003

1. Properties of Perspective projection 4 hours.

Aim:
Difference between perspective projection and orthogonal projection

Theory:
Coordinate systems, inner and outer orientation
Mathematical model of co linearity equation

The essential of photogrammetry technique is the possibility to obtain the information about 3D objects by their 2D images. In the initial development of photogrammetry the obtained images are photo images, but now the methods for registration are very different. The images are registered not only in analogue form by photochemical processes, but also in digital form by using of optical-mechanical system, electron- mechanical systems (push broom scanners), by radar systems or by laser systems, or by thermal systems. In traditional photogrammetric system the main principal of geometrical transformation is central projection that in mathematical sense means perspective projection.

1.1. Mathematical foundations

The analytical presentation is based on the space transformation between object co-ordinates from space co-ordinate system to co-ordinate in representing systems, called mapping co-ordinate system. The co-ordinate systems are orthogonal so the transformation does not change the angle between lines and by this reason is call similarity transformation.

1.1.1. Space similarity transformation

The space transformation could be separated into three-plane transformation. Every one of these transformations is in the plane of the two coordinate axes. In photogrammery the rotation angles are defined as follows:

- φ in xz plane;
- ω in yz plane;
- κ in xy plane.
-

For the rotation in plane the simple relation can be derived based on figure 1.1.

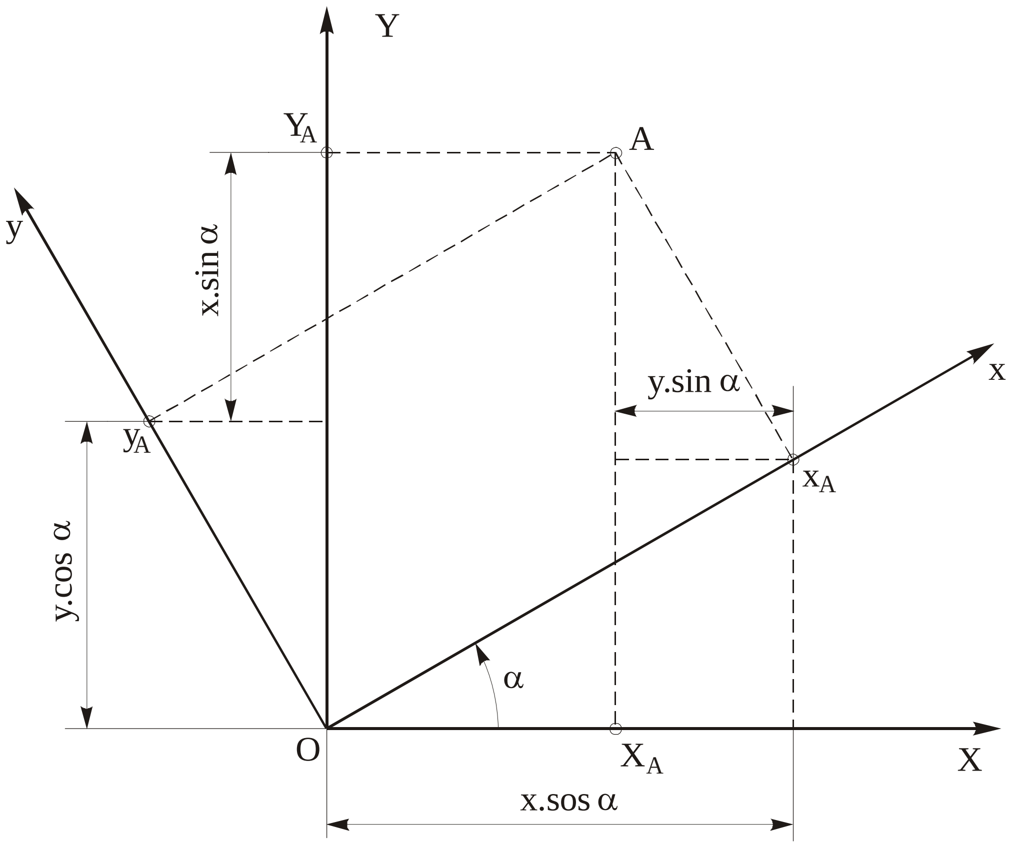

Figure 1.1. Rotation in plane

The relation between coordinates of base coordinate system XY and rotated xy are

$$\begin{aligned} X &= x.\cos\alpha - y.\sin\alpha \\ Y &= y.\sin\alpha + x.\cos\alpha \end{aligned} \tag{1.1}$$

In this case the positive direction of rotation is counterclockwise. The signs before **sin** terms depend on the positive direction of rotation angles.

The rotation matrixes are as follows:

The transformation matrix for rotation about x axes in counterclockwise direction on angle ω is:

$$R_\omega = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{bmatrix} \tag{1.2}$$

Transformation could be defined

$$U = R_\omega u_{\varphi\kappa} \qquad \begin{bmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{bmatrix} = R_\omega . \begin{bmatrix} x_{\varphi\kappa} \\ y_{\varphi\kappa} \\ z_{\varphi\kappa} \end{bmatrix} \tag{1.3}$$

The transformation about y axes of $u_{\varphi\kappa}$ co-ordinate system is clockwise on angle φ. The rotation matrix is:

$$R_{\varphi} = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix} \qquad (1.4)$$

The transformation is defined as follows

$$u_{\varphi\kappa} = R_{\varphi}.u_{\kappa} \qquad \begin{bmatrix} \mathrm{x}_{\varphi\kappa} \\ \mathrm{y}_{\varphi\kappa} \\ \mathrm{z}_{\varphi\kappa} \end{bmatrix} = R_{\varphi}.\begin{bmatrix} x_{k} \\ y_{\kappa} \\ z_{\kappa} \end{bmatrix} \qquad (1.5)$$

The resulting matrix derived from multiplication of two rotation matrixes could be defined as

$$R_{\omega\varphi} = R_{\omega}.R_{\varphi} \ . \qquad (1.6)$$

The transformation for object co-ordinates is

$$U = R_{\omega\varphi}.u_{\kappa} \qquad (1.7)$$

By analogy could be defined the rotation about z axes of the co-ordinate system u_{κ} . It is on angle κ in counter clockwise direction.

$$R_{\kappa} = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (1.8)$$

For the transformation could be written

$$u_{\kappa} = R_{\kappa}.u$$
$$\boldsymbol{U} = R_{\omega}.R_{\varphi}.R_{\kappa}.\boldsymbol{u} \quad \boldsymbol{U} = \boldsymbol{R}.\boldsymbol{u} \qquad (1.9)$$

where

$$R_{\omega\varphi\kappa} = R_{\omega}.R_{\varphi}.R_{\kappa} \qquad (1.10)$$

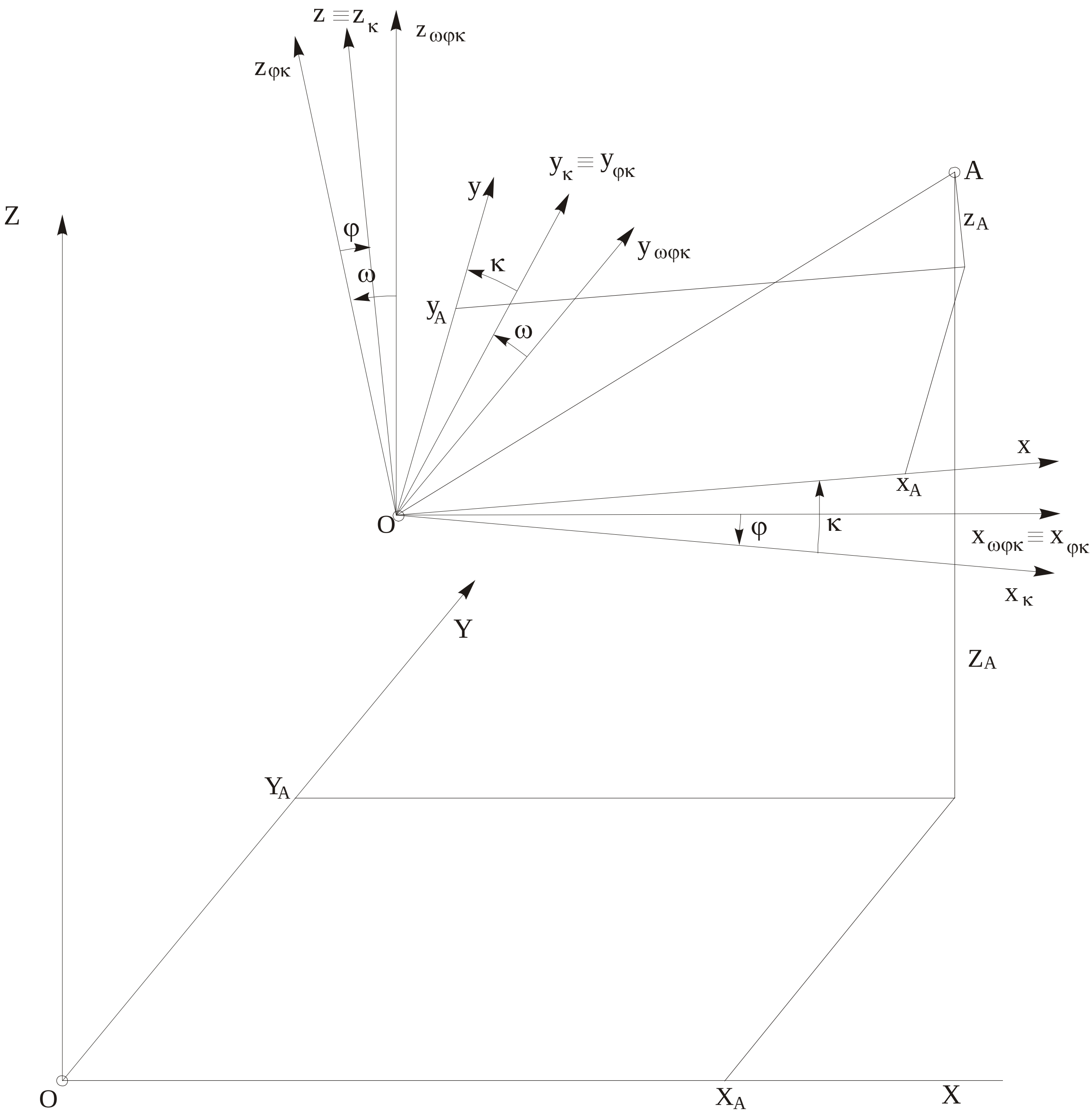

Figure 1.2. Rotation in space

The result is sensitive to the sequence of multiplication. If the sequence is changed we obtain another result. For the rotation matrix $R_{\varphi\omega\kappa}$ we obtain

$$R_{\varphi\omega\kappa} = R_{\varphi}.R_{\omega}.R_{\kappa} \tag{1.11}$$

The coefficients of these two matrixes are shown in table 1.1

Table 1.1

r_{ij}	$R_{\omega\varphi\kappa}$	$R_{\varphi\omega\kappa}$
r_{11}	$\cos\varphi.\cos\kappa$	$\cos\varphi.\cos\kappa+\sin\varphi.\sin\omega.\sin\kappa$
r_{12}	$-\cos\varphi.\sin\kappa$	$-\cos\varphi.\sin\kappa+\sin\varphi.\sin\omega.\cos\kappa$
r_{13}	$\sin\varphi$	$\sin\varphi.\cos\omega$
r_{21}	$\cos\omega.\sin\kappa+\sin\omega.\sin\varphi.\cos\kappa$	$\cos\omega.\sin\kappa$
r_{22}	$\cos\omega.\cos\kappa-\sin\omega.\sin\varphi.\sin\kappa$	$\cos\omega.\cos\kappa$
r_{23}	$-\sin\omega.\cos\varphi$	$-\sin\omega$
r_{31}	$\sin\omega.\sin\kappa-\cos\omega.\sin\varphi.\cos\kappa$	$-\sin\varphi.\cos\kappa+\cos\varphi.\sin\omega.\sin\kappa$
r_{32}	$\sin\omega.\cos\kappa+\cos\omega.\sin\varphi.\sin\kappa$	$\sin\varphi.\sin\kappa+\cos\varphi.\sin\omega.\cos\kappa$
r_{33}	$\cos\omega.\cos\varphi$	$\cos\varphi.\cos\omega$

The rotation angles could be derived from rotation matrixes. The angles obtained from the matrix $R_{\varphi\omega\kappa}$ have the following form:

$$\tan\varphi=\frac{r_{13}}{r_{33}},\qquad \sin\omega=-r_{23},\qquad \tan\kappa=\frac{r_{21}}{r_{22}} \tag{1.12}$$

As the rotation matrixes are orthogonal then the inverse matrix is equal to the transposed one.

$$\boldsymbol{R}^{-1}=\boldsymbol{R}^{t} \tag{1.13}$$

The inverse solution is obtained easy

$$\boldsymbol{u}=\boldsymbol{R}^{t}_{\varphi\omega\kappa}.\boldsymbol{U}=\begin{bmatrix} r_{11} & r_{21} & r_{31}\\ r_{12} & r_{22} & r_{32}\\ r_{13} & r_{23} & r_{33}\end{bmatrix}.\begin{bmatrix} X\\ Y\\ Z\end{bmatrix} \tag{1.14}$$

The matrix R is called sometimes the matrix of cosine directories because its columns are the vectors of the rotated coordinate system in the basic (object) co-ordinate system.

$$R=\begin{bmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\end{bmatrix}=\begin{bmatrix} i_x & j_x & k_x\\ i_y & j_y & k_y\\ i_z & j_z & k_z\end{bmatrix} \tag{1.15}$$

If the translation is added to the rotation and the possibility for scaling is defined we obtained the main form of 3D transformation.

$$\boldsymbol{U}_i = \boldsymbol{U}_0 + m.\boldsymbol{R}.\boldsymbol{u}_i \tag{1.16}$$

This equation allows to define the conformal transformation tat is called similarity transformation. For this type transformation all points equal scaling factor. The points in object space and mapping (modeling) space has their 3D co-ordinates and the shape of the object is preserved. The mathematical formulation of similarity transformation could be used for definition of projective transformation. For projective transformation all points lie on the same plane in the transformed (modeling) space. In such case the z co-ordinates are the same (-c). The orthogonal ray from projection center O to the projection plane is called perspective axis. It intersects the projection plane in principle point p. In the projection plane is defined co-ordinate system ξη with origin M. The principle point has coordinates $\boldsymbol{p} = (\xi_0, \eta_0)$. In photogrammetry the co-ordinate system in projection plane is defined by fiducial marks. Finally the equation obtain the form

$$\boldsymbol{U}_i = \boldsymbol{U}_0 + m_i.\boldsymbol{R}.(p_i - p_p) \tag{1.17}$$

$$\text{where } p_i = \begin{bmatrix} \xi_i \\ \eta_i \\ 0 \end{bmatrix}, \quad p_p = \begin{bmatrix} \xi_p \\ \eta_p \\ c \end{bmatrix} \tag{1.18}$$

The distance Op between projection center and principle point is called principle distance and usually is denoted by *c*.

1.1.2. Colinearity equations

It is important to mention the main feature of projective transformation of 3D point over the plane. The information is lost so it is not possible to restitute the space position of the point by its plane image. For this purposes could be used additional information like that point lie on plane or line or to used second image from another projection center that is main essence of stereo photogrammetry. The transformation equation could be solved and three equations with one undefined parameter m_i are obtained.

$$\boldsymbol{p}_i - \boldsymbol{p}_p = \tfrac{1}{m_i}\boldsymbol{R}^t.(\boldsymbol{U}_i - \boldsymbol{U}_0) \tag{1.19}$$

In scalar form the equations are as follows

$$\begin{aligned} x - x_p &= \tfrac{1}{m_i}\left[r_{11}.(X_i - X_0) + r_{21}.(Y_i - Y_0) + r_{31}.(Z_i - Z_0)\right] \\ y - y_p &= \tfrac{1}{m_i}\left[r_{12}.(X_i - X_0) + r_{22}.(Y_i - Y_0) + r_{32}.(Z_i - Z_0)\right] \\ -c &= \tfrac{1}{m_i}\left[r_{13}.(X_i - X_0) + r_{23}.(Y_i - Y_0) + r_{33}.(Z_i - Z_0)\right] \end{aligned} \tag{1.20}$$

Substituting the scale factor from the third equation in first two we obtain the analytical form of colinearity equations.

$$x - x_p = -c.\frac{r_{11}.(X_i - X_0) + r_{21}.(Y_i - Y_0) + r_{31}.(Z_i - Z_0)}{r_{13}.(X_i - X_0) + r_{23}.(Y_i - Y_0) + r_{33}.(Z_i - Z_0)}$$
$$y - y_p = -c.\frac{r_{12}.(X_i - X_0) + r_{22}.(Y_i - Y_0) + r_{32}.(Z_i - Z_0)}{r_{13}.(X_i - X_0) + r_{23}.(Y_i - Y_0) + r_{33}.(Z_i - Z_0)} \quad (1.21)$$

This form of colinearity equations corresponds to the aerial photos in photogrammetry. Making some simplifications in colinearity equations it is possible to derived some main features of projective transformation. For points on the XY plane we can define

$$Z_i = 0, \quad X_0 = 0, \quad \mathrm{Y}_0 = 0 \quad (1.22)$$

After some transformation it is possible to reach the well known form of projective transformation in plane.

$$x_i - x_p = \tfrac{1}{m}.\frac{a'_x.X_i + b'_x.Y_i + c'_x}{d.X_i + e.Y_i + 1}$$
$$y_i - y_p = \tfrac{1}{m}.\frac{a'_y.X_i + b'_y.Y_i + c'_y}{d.X_i + e.Y_i + 1}, \quad (1.23)$$

where the parameters are as follows:

$$m = \frac{Z_0}{c}, \quad d = \frac{r_{13}}{r_{33}.Z_0}, \quad e = \frac{r_{23}}{r_{33}.Z_0}$$
$$a'_x = \frac{r_{11}}{r_{33}}, \quad b'_x = \frac{r_{21}}{r_{33}}, \quad c'_x = \frac{r_{31}}{r_{33}} \quad . \quad (1.24)$$
$$a'_y = \frac{r_{12}}{r_{33}}, \quad b'_y = \frac{r_{22}}{r_{33}}, \quad c'_y = \frac{r_{32}}{r_{33}}$$

It is possible to convert equations (1.23) in the wellknown form

$$x_i = \frac{a_x.X_i + b_x.Y_i + c_x}{d.X_i + e.Y_i + 1}$$
$$y_i = \frac{a_y.X_i + b_y.Y_i + c_y}{d.X_i + e.Y_i + 1} \quad (1.25)$$

where the parameters are

$$a_x = \frac{a'_x}{m} + \mathrm{x_p.d} \quad \mathrm{b_x} = \frac{a'_y}{m} + \mathrm{x_p.e} \quad \mathrm{c_x} = \frac{\mathrm{c'_x}}{\mathrm{m}} + x_p$$
$$a_y = \frac{a'_x}{m} + \mathrm{y_p.d} \quad \mathrm{b_y} = \frac{a'_y}{m} + \mathrm{y_p.e} \quad \mathrm{c_x} = \frac{\mathrm{c'_x}}{\mathrm{m}} + y_p \tag{1.26}$$

From the relations for projective transformation could be derived the equations for parallel projection. In the case of parallel projection the projection center is in infinity. The principle distance and Z_0 tends to infinity.

$$c \to \infty, \qquad Z_0 \to \infty$$
$$m = 1, \quad d = 0, \quad e = 0 \tag{1.27}$$

The equations are transformed to equations of affine transformation

$$x_i = a_x.X_i + b_x.Y_i + c_x$$
$$y_i = a_y.X_i + b_y.Y_i + c_y \tag{1.28}$$

From the equations for affine transformation and for the projective transformation is possible to obtain some features of perspective projection and of parallel projection.

For the case of parallel projection three points lying on the same line are connected with harmonic relation. To derive this feature it is possible to use lines parallel to coordinate axes.

We define $X_i = 0$

For three points we obtain

$$x_1 = b_x.Y_1 + c_x \quad y_1 = b_y.Y_1 + c_y$$
$$x_2 = b_x.Y_2 + c_x \quad y_2 = b_y.Y_2 + c_y \tag{1.29}$$
$$x_3 = b_x.Y_3 + c_x \quad y_3 = b_y.Y_3 + c_y$$

Distances between points over the co-ordinate axes are as follows

$$x_2 - x_1 = b_x.(Y_2 - Y_1) \quad y_2 - y_1 = b_y.(Y_2 - Y_1)$$
$$x_3 - x_2 = b_x.(Y_3 - Y_2) \quad y_3 - y_2 = b_y.(Y_3 - Y_2) \tag{1.30}$$

The ratios between lengths are the same

$$\frac{x_3 - x_2}{x_2 - x_1} = \frac{Y_3 - Y_2}{Y_2 - Y_1}$$
$$\frac{y_3 - y_2}{y_2 - y_1} = \frac{Y_3 - Y_2}{Y_2 - Y_1} \tag{1.31}$$

The results for X axes are the same. This leads to relations

$$\lambda = \frac{BC}{AB} = \frac{bc}{ab} \tag{1.32}$$

This relation corresponds to the case of parallel projection. It is shown graphically on the next figure.

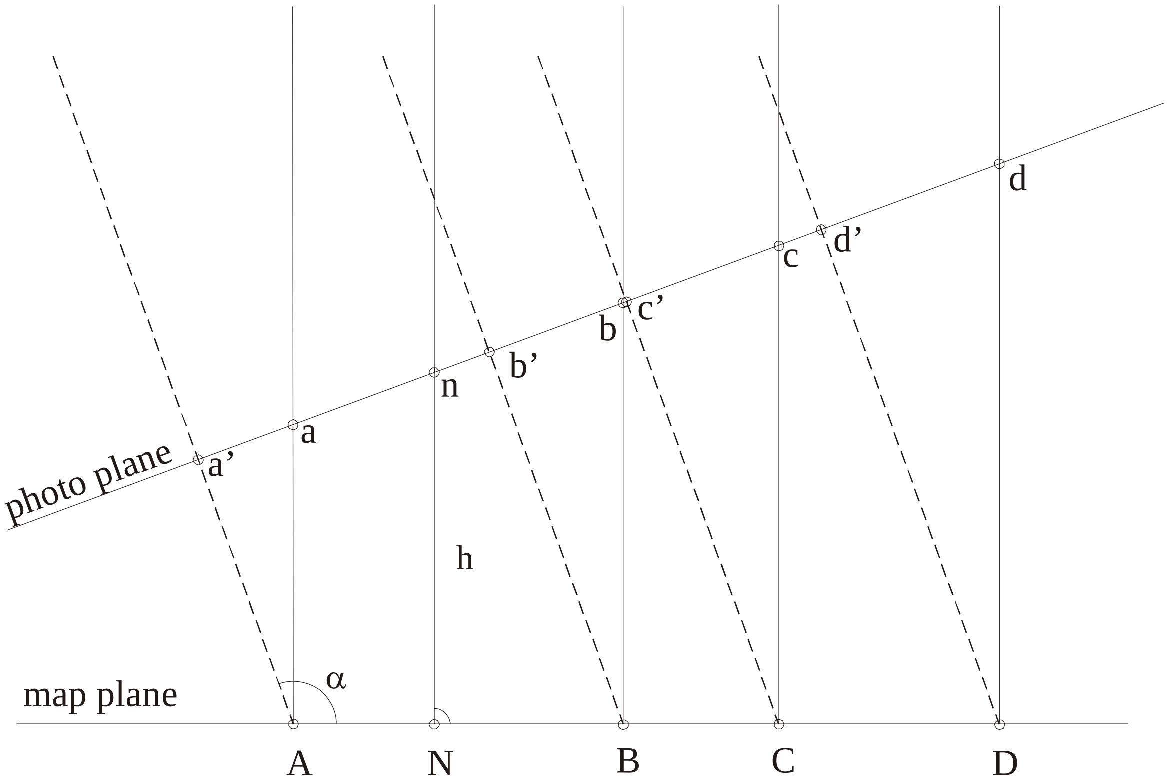

Figure 1.3. Parallel projection between map and image plane

1.2. Projective Transformation

1.2.1. Mathematical formulation

The analogue results could be obtained for perspective projection. But in this case the used relation is connected four points and it is known as anharmonic cross-ratio or cross-ratio of projective geometry.

$$\lambda = (p_1, p_2, p_3, p_4) = \frac{p_1 p_3}{p_2 p_3} : \frac{p_1 p_4}{p_2 p_4} \tag{1.33}$$

The one-dimensional derivation could be performed from the co-ordinate relation of projective transformation. We set $X_i = 0$. For co-ordinates of four points we obtain the relations

$$x_1 = \frac{a_x.X_1 + c_x}{d.X_1 + 1} \quad y_1 = \frac{a_y.X_1 + c_y}{d.X_1 + 1}$$
$$x_2 = \frac{a_x.X_2 + c_x}{d.X_2 + 1} \quad y_2 = \frac{a_y.X_2 + c_y}{d.X_2 + 1}$$
$$x_3 = \frac{a_x.X_3 + c_x}{d.X_3 + 1} \quad y_3 = \frac{a_y.X_3 + c_y}{d.X_3 + 1} \tag{1.34}$$
$$x_4 = \frac{a_x.X_4 + c_x}{d.X_4 + 1} \quad y_4 = \frac{a_y.X_4 + c_y}{d.X_4 + 1}$$

We shall show the calculation of first distance $p_1 p_3$

$$x_3 - x_1 = \frac{a_x.X_3 + c_x}{d.X_3 + 1} - \frac{a_x.X_1 + c_x}{d.X_1 + 1} =$$
$$= \frac{(a_x.X_3 + c_x).(d.X_1 + 1) - (a_x.X_1 + c_x).(d.X_3 + 1)}{(d.X_1 + 1).(d.X_3 + 1)} =$$
$$= \frac{a_x.d.(X_3 X_1 - X_1 X_3) + a_x.(X_3 - X_1) + c_x.d.(X_1 - X_3) + c_x - c_x}{(d.X_1 + 1).(d.X_3 + 1)} = \tag{1.35}$$
$$= \frac{(a_x - c_x.d).(X_3 - X_1)}{(d.X_1 + 1).(d.X_3 + 1)}$$

In the same way it is possible to obtain the distances in x co-ordinate for the fourth combinations of points

$$x_3 - x_1 = \frac{(a_x - c_x.d).(X_3 - X_1)}{(d.X_1 + 1).(d.X_3 + 1)}$$
$$x_3 - x_2 = \frac{(a_x - c_x.d).(X_3 - X_2)}{(d.X_2 + 1).(d.X_3 + 1)}$$
$$x_4 - x_1 = \frac{(a_x - c_x.d).(X_4 - X_1)}{(d.X_1 + 1).(d.X_4 + 1)} \tag{1.36}$$
$$x_4 - x_2 = \frac{(a_x - c_x.d).(X_4 - X_2)}{(d.X_2 + 1).(d.X_4 + 1)}$$

From distances it is possible to construct the anharmonic ratio λ

$$\lambda = \frac{x_3 - x_1}{x_3 - x_2} : \frac{x_4 - x_1}{x_4 - x_2} =$$

$$= \frac{\dfrac{(a_x - c_x.d).(X_3 - X_1)}{(d.X_1 + 1).(d.X_3 + 1)}}{\dfrac{(a_x - c_x.d).(X_3 - X_2)}{(d.X_2 + 1).(d.X_3 + 1)}} : \frac{\dfrac{(a_x - c_x.d).(X_4 - X_1)}{(d.X_1 + 1).(d.X_4 + 1)}}{\dfrac{(a_x - c_x.d).(X_4 - X_2)}{(d.X_2 + 1).(d.X_4 + 1)}} = \quad (1.37)$$

$$= \frac{X_3 - X_1}{X_3 - X_2} : \frac{X_4 - X_1}{X_4 - X_2}$$

The results obtained for distances over y axes are similar. Finally this proves the validity of equality of anharmonic ratio in object plane and in the modeling plane.

$$\lambda = \frac{p_1 p_3}{p_2 p_3} / \frac{p_1 p_4}{p_2 p_4} = \frac{P_1 P_3}{P_2 P_3} / \frac{P_1 P_4}{P_2 P_4} \quad (1.38)$$

For photogrammetric purposes the this relations are for photo plane and map plane

1.2.2. Application in photogrammetry

The one dimensional cross ratio for projective case is shown on the following figure.

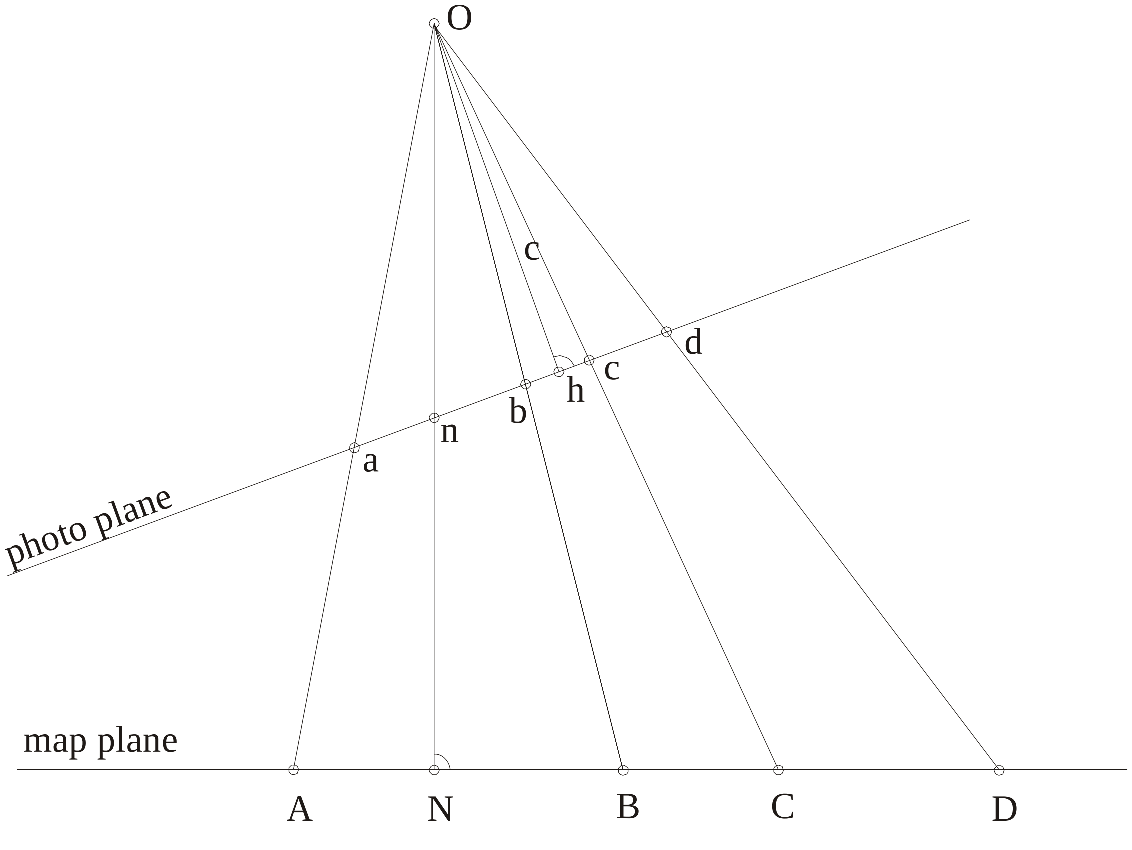

Figure 1.4. Cross ratio

The cross-ratio for four points in image plane could be derived by different presentations of the areas of the corresponding triangles.

$$
\begin{aligned}
(abcd) &= \frac{ac}{bc}:\frac{ad}{bd} = \frac{ac}{bc}.\frac{bd}{ad} = \\
&= \frac{\tfrac{1}{2}Op.ac}{\tfrac{1}{2}Op.bc}.\frac{\tfrac{1}{2}Op.bd}{\tfrac{1}{2}Op.adc} = \frac{\tfrac{1}{2}aO.cO.\sin<aOc}{\tfrac{1}{2}bO.cO.\sin<bOc}.\frac{\tfrac{1}{2}bO.dO.\sin<bOd}{\tfrac{1}{2}aO.dO.\sin<aOd} = \\
&= \frac{\sin<aOc}{\sin<bOc}.\frac{\sin<bOd}{\sin<aOd}
\end{aligned}
\qquad (1.39)
$$

For anharmonic cross-ratio in the map plane the same value is obtained

$$
(ABCD) = \frac{AC}{BC}:\frac{BC}{BD} = \frac{\sin<AOC}{\sin<BOC}.\frac{\sin<BOD}{\sin<AOD} \qquad (1.40)
$$

The fundamental theorem of projective geometry follows from invariance of anharmonic cross-ratio. It states that the projection is defined by three couples of homologuos points lying on the three rays from the projective center. The position of the fourth point could be determined by the cross-ratio equality.

1.2.3. Projective relationship

The main definitions of central projection are derived from the relationship between map and image. They are presented on the figure 1.2. The quadrilateral in photo plane is projected trough projection center *O* over the map plane into quadrilateral *ABCD*. The elevation of projection center h is measured along the plumb line *ON*. The photo principal distance c is along the photograph perpendicular *O*p, where p is the principal point in the image plane. The lines *ON* and *Op* determine the principal plane whose traces np and *NP* are principal line in the photo- and map-planes respectively. The tilt γ of the photo-plane is <pOn. The same value has the angle <pSP between photo- and map-planes at their intersection, which is the axes of perspective. Isocenters *i* and I are the crossing points of the line in the principal plane, which bisects angle <nOp. The point *i* is the point with minimal deformations. A horizontal plane trough *O* intersects photo plane in the photo vanishing line or he true horizon. The projection of this line on the map plane is at infinity (i.e. intersection of parallel planes). The photo vanishing point V is at the intersection of photo vanishing line and the principal plane. Point V is equidistant from projection center *O* and isocenter *i*. A plane parallel to the photo plane trough *O* intersects map plane at the map vanishing line (the line projected from the infinity on the photo plane). The map vanishing point W is at the intersection of map vanishing line and the principal plane. The point *W* is equidistant from projection center *O* and the isocenter *I* in the map plane.

It is important to note that all this definitions are synonymous if angle γ is not 0. If it is zero the photo plane and map plane are parallel and all these intersection lines are in the infinity.

For example, the parallel projections onto the map plane for lines, converging in photo plane into the point lying over the photo vanishing line, in map plane are parallel. If such parallel projection lines are parallel to the principal line in map plane, their homologues lines converge in the point V. The projections onto the map plane of all parallel lines in photo plane converge in map plane in the same point onto the map vanishing line. If such lines are parallel to principal line in photo plane, their projections converge in the point W. However if both planes (photo- and map-) are parallel, then the projection lines rest parallel too.

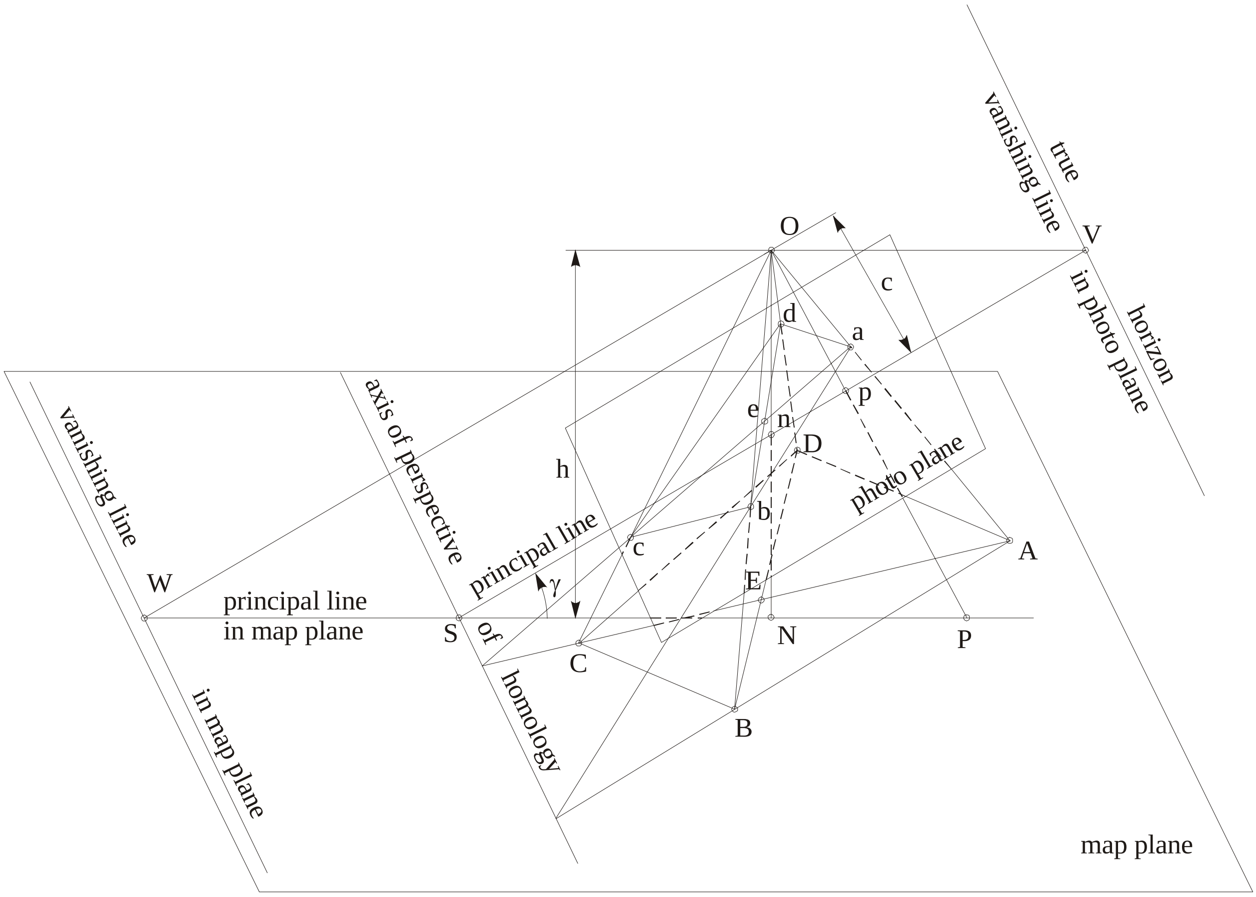

Figure 1.5. Projective relationships

1.2.4. Projective relationship between photography and ground

The relationships between map and photo correspond to the relations between point lying on the photo and points lying on the horizontal plane. In practical cases when every point has different height the displacement of image points' position occurs. Such displacement cold be shown graphically on the figure 1.4.

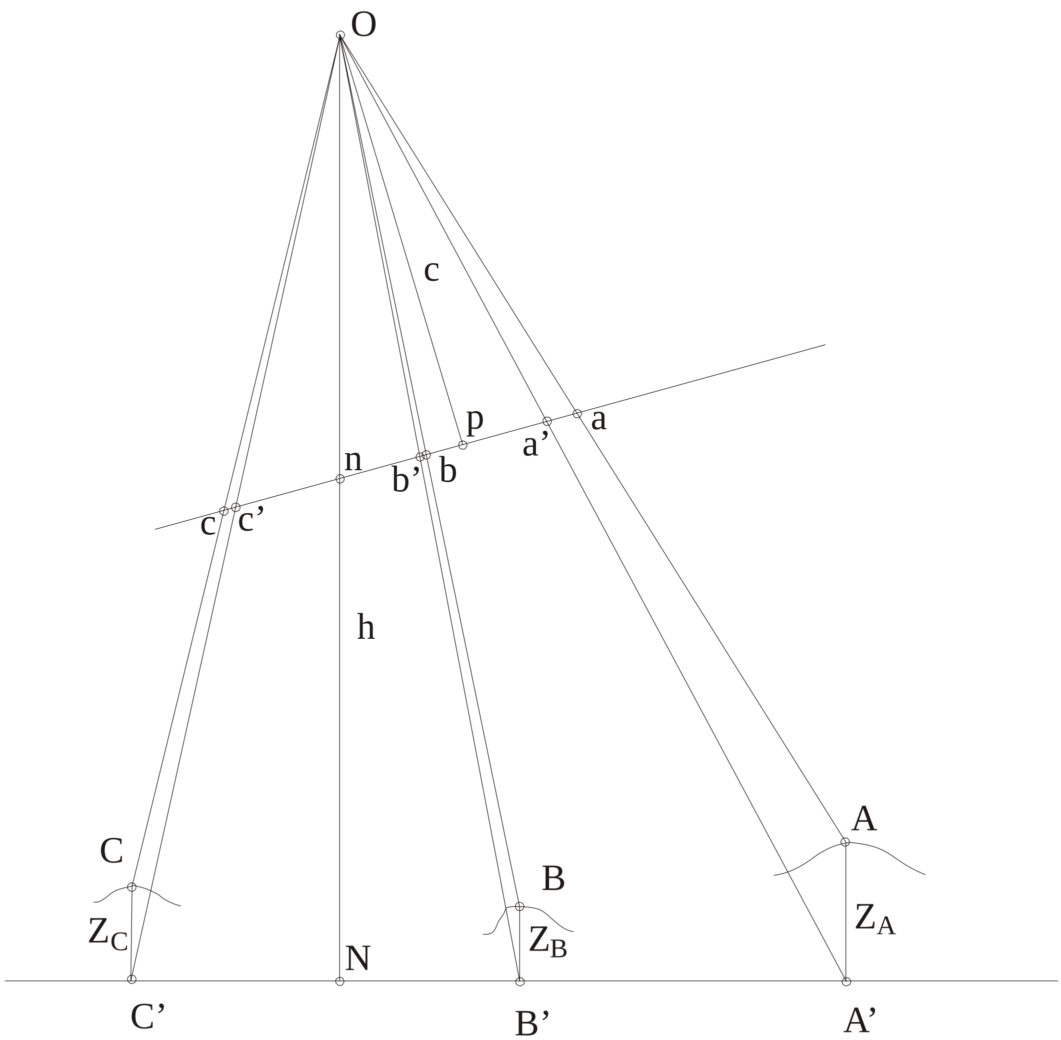

Figure 1.6. Projective relationship in case of ground heights’ influence

The shown displacements are in the case where all points lie in the principle plane. In common case the displacement of the points lies on the intersection of photo plane with the plane defined by plumb line trough O and vertical line trough the ground point.

By the same reason the vertical edges of the buildings lie over lines, passing trough the image n of the nadir point N.

1.3. Co-ordinate systems in photogrammetry

For working in photogrammetry there are defined several coordinate systems

1.3.1. Image coordinate system

Image coordinate system is defined by the coordinate system of fiducial marks of image. The x axis of image coordinate system coincides with straight line between horizontal marks and its center is the middle of the line. This system is connected with coordinate system of the camera. The camera coordinate system has the same orientation as image coordinate system but its center lies at distance c (camera constant) from the image plane. The perpendicular line through its center O crosses the image plane in principal point P. Principal point p has coordinates (ξ_p, η_p)

Geometrically these relations are shown on figure 1.7.

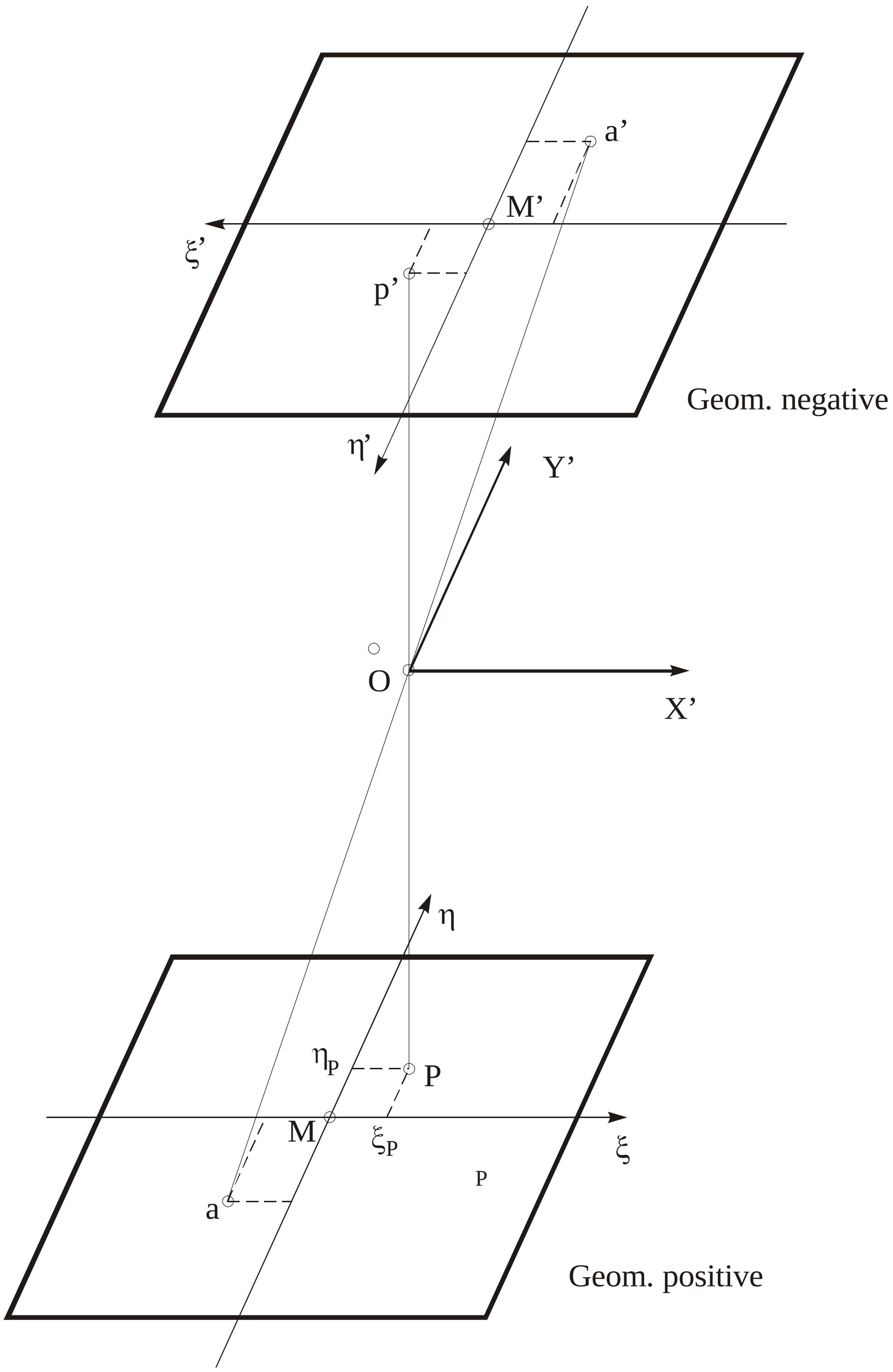

Figure 1.7. Image coordinate system

The image in camera position is geometrically negative. The image in projector position is geometrically positive. Practically this is corresponding of observing of photo negative from emulsion side (geometrically negative or mirror image) or observing the negative photo from the base side (geometrically positive non-mirror). If the contact copy (emulsion to emulsion) is made then it is photographic positive and from emulsion side it is geometrically positive but from back side (if base is transparent) it is geometrically negative.

Space image coordinate system has the same orientation as plane image coordinate system but passes through the projection center of camera.

Its' coordinates can be defined by relation

$$x' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \xi - \xi_p \\ \eta - \eta_p \\ -c \end{bmatrix} \tag{1.41}$$

1.3.2. Comparator coordinate system

Comparator coordinate system is the coordinate system of the photogrammetric apparatus on which the image coordinates are measured.

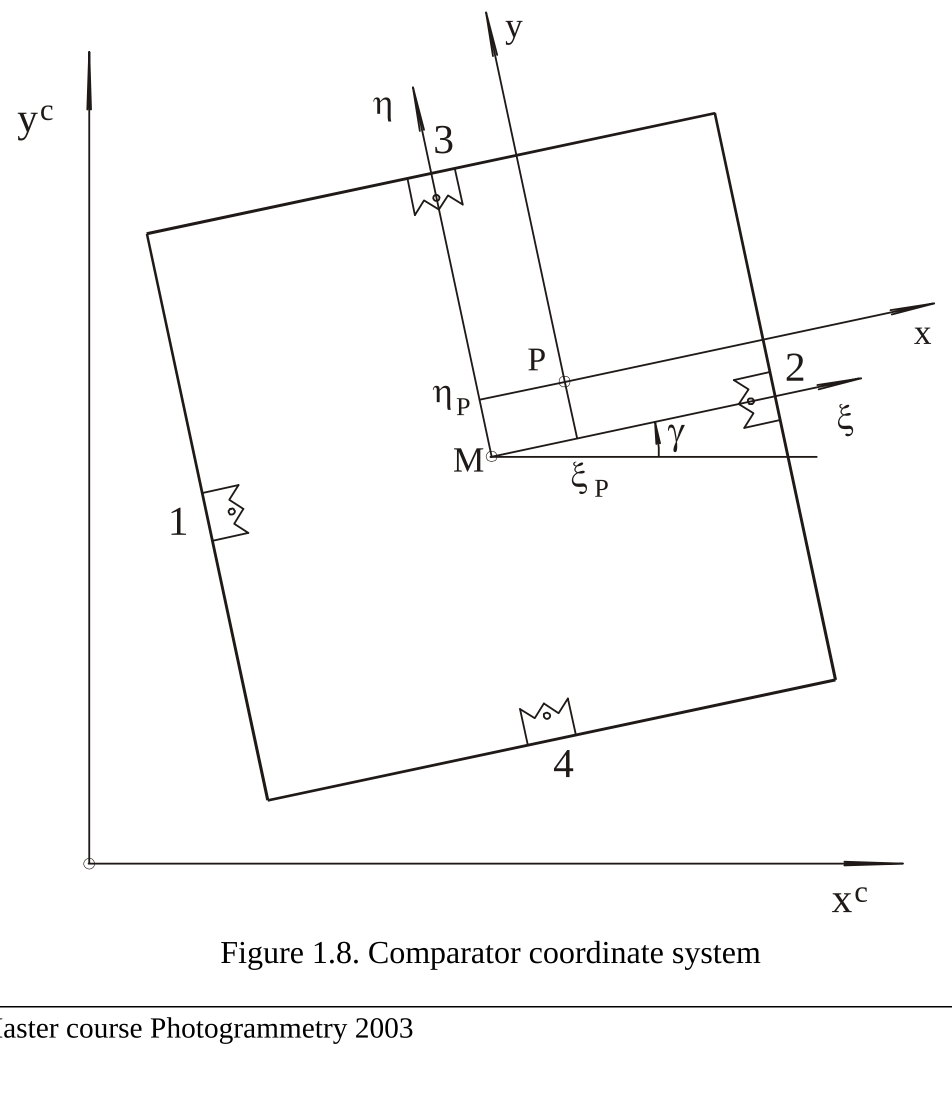

Figure 1.8. Comparator coordinate system

1.3.3. Model coordinate system

Model coordinate system is orthogonal space coordinate system that is usually connected with stereopair.

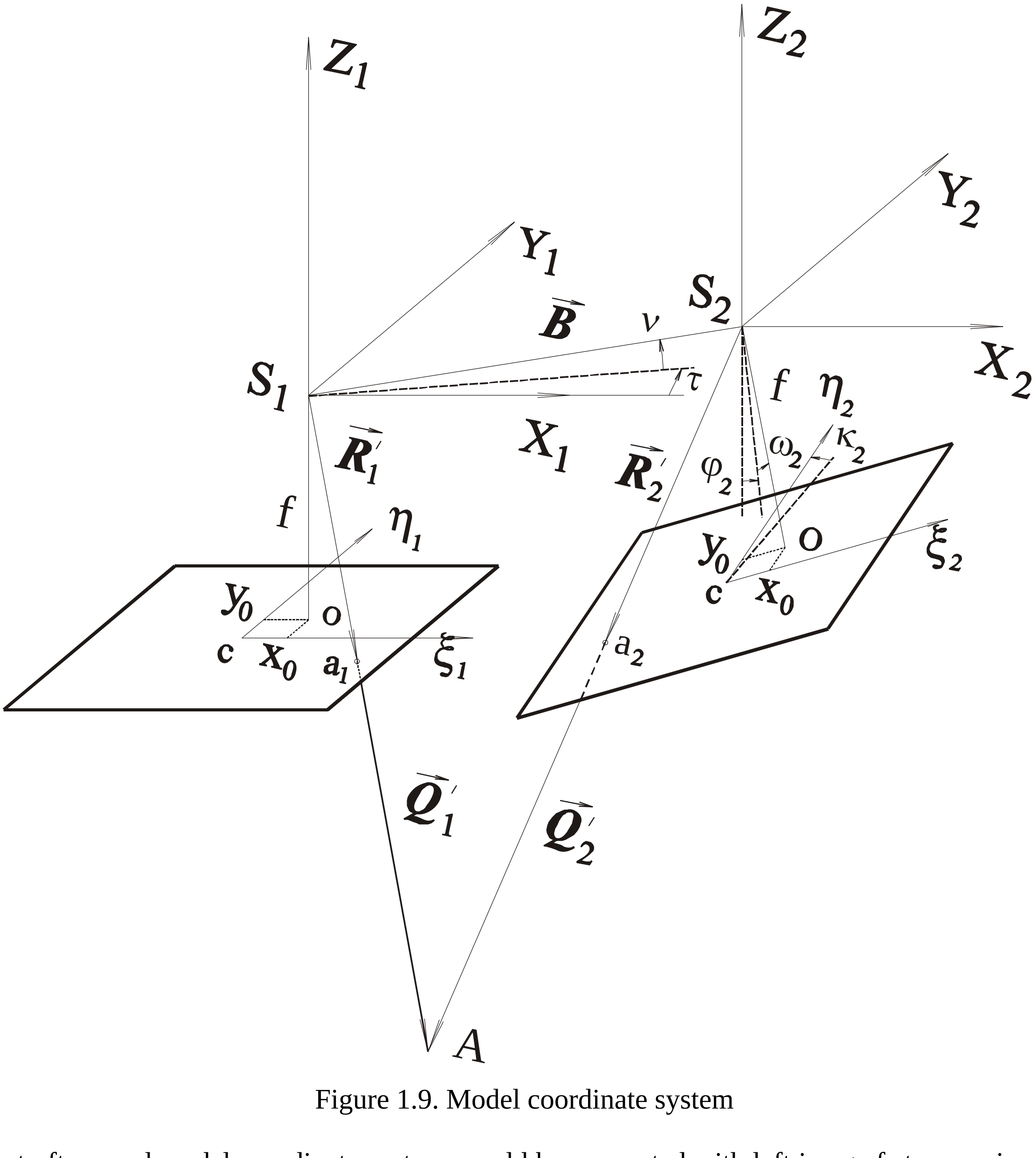

Figure 1.9. Model coordinate system

Most often used model coordinate systems could be connected with left image f stereo pair, or to be defined by the direction of base and the principal direction of left photo.

1.3.4. Object coordinate system

Object coordinate system is orthogonal (Cartesian) coordinate system that is connected with the terrain or processed object. In the cases of aerial or space photogrammetry this coordinate system is connected with global geodetic coordinate systems. Such systems are geocentric and Geocentric Local Vertical. Both coordinate systems are Cartesian.

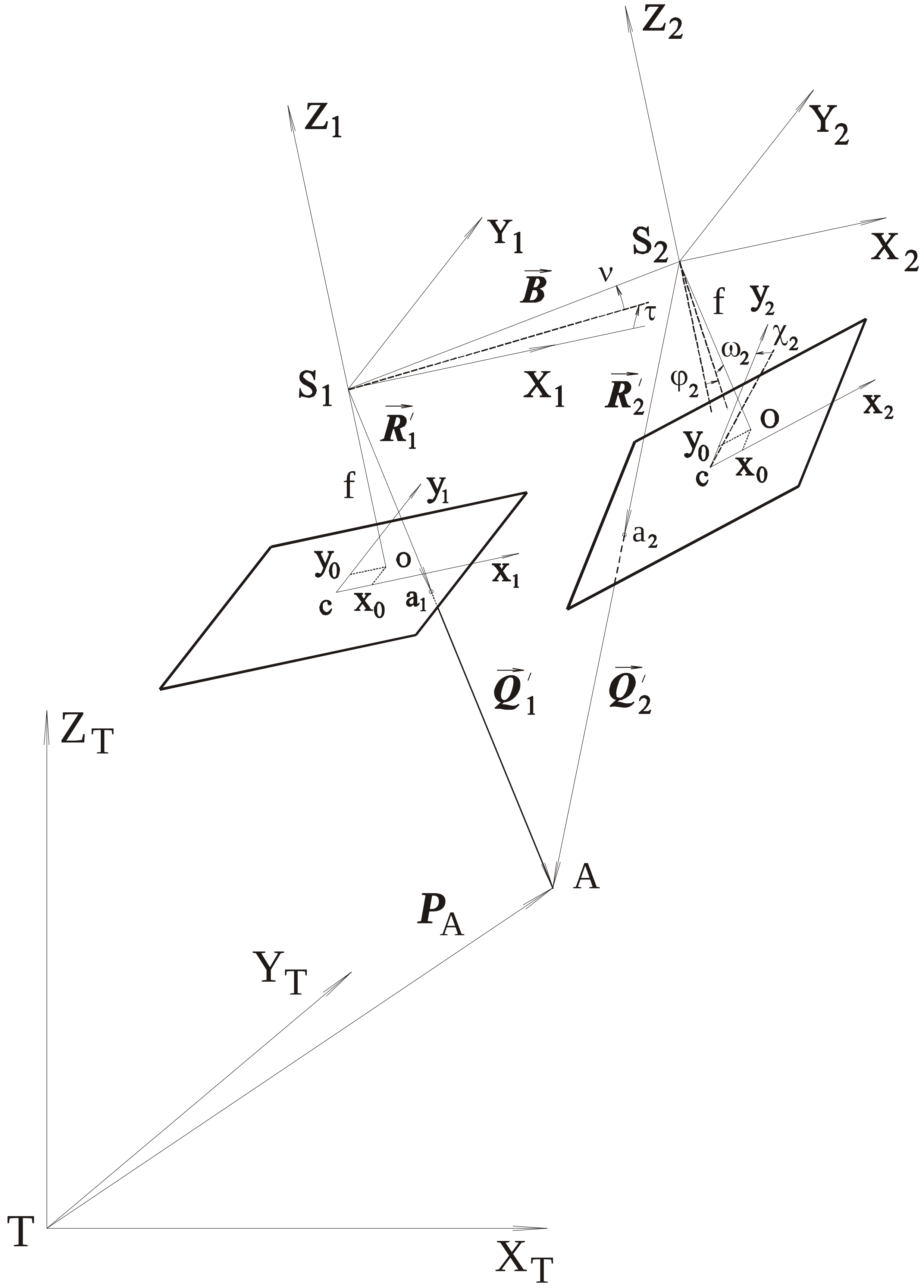

Figure 1.10. Object coordinate system

It is important to be emphasize that all computations in photogrammetry must be done in orthogonal coordinate system.

1.4. Inner (interior) orientation

The interior orientation of the photo refers to the perspective geometry of the camera. The parameters of interior orientation are c-ordinates of vector $\overrightarrow{PO}$, which determines the position of projection center respectively to the center of co-ordinate system of the photo. Another parameters are distortion parameters of the camera, which are determined in the process of camera calibration.

The co-ordinate system of the photo is defined usually by the axis through horizontal fiducial marks A-C. The y axis is perpendicular to x axis and passes trough center f co-ordinate system O'. This center must coincide with the principal point of the photo, but due to the imperfectness of camera they differ. The co-ordinates of principle point are (x_p, y_p).

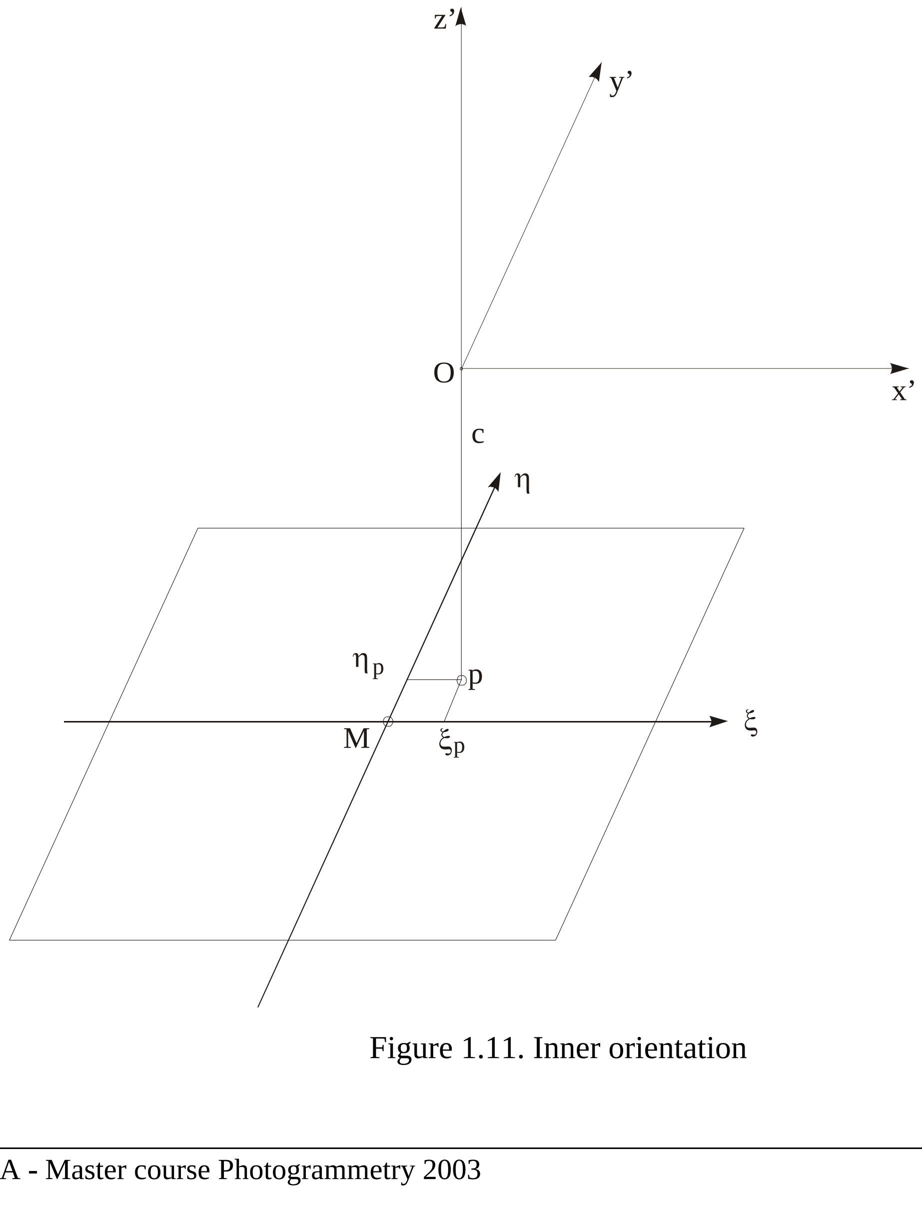

Figure 1.11. Inner orientation

For inner orientation usually are used the measured and calibrated values of fiducial marks. For transformation could be used projective transformation, affine transformation or similarity transformation. The application of projective transformation des not gives the possibility for equalization by means of least square method, the transformation of lower order are applied.

$$\begin{aligned}\xi_i &= a_x.x_i^c + b_x.y_i^c + c_x - \xi_p \\ \eta_i &= a_y.x_i^c + b_y.y_i^c + c_y - \eta_p\end{aligned} \qquad (1.42)$$

If the co-ordinates of fiducial marks are given relatively to the principle point then the values of ξ_p, η_p are included in the coefficients c_x, c_y. If suggestion for similarity transformation is made then

$$\begin{aligned}a_x &= b_y \\ b_x &= -a_y\end{aligned} \qquad (1.43)$$

The transformation equations are converted to the form

$$\begin{aligned}\xi_i &= a.x_i^c - b.y_i^c + c_x \\ \eta_i &= b.x_i^c + a.y_i^c + c_y\end{aligned} \qquad (1.44)$$

In some simple cases the scaling of measured coordinates is not made so the coefficients a and b take the form

$$\begin{aligned}a &= \cos\gamma \\ b &= \sin\gamma\end{aligned} \qquad (1.45)$$

In this case the transformation has the form

$$\begin{aligned}\xi_i &= \sqrt{1-b^2}.x_i^c - b.y_i^c + c_x \\ \eta_i &= b.x_i^c + \sqrt{1-b^2}.y_i^c + c_y\end{aligned} \qquad (1.46)$$

This type of transformation is applied very rarely.

There are possible polynomial and bilinear transformations. They are applied for reseau cameras or for CCD cameras.

The equations for polynomial transformation have the form

$$\begin{aligned}\xi &= a_{00} + a_{10}.x^c + a_{11}.y^c + a_{20}.x^{c2} + a_{21}.x^c.y^c + a_{22}.y^{c2} \\ \eta &= b_{00} + b_{10}.x^c + b_{11}.y^c + b_{20}.x^{c2} + b_{21}.x^c.y^c + b_{22}.y^{c2}\end{aligned} \qquad (1.47)$$

Bilinear transformation is similar to affine but it has one coefficient more

$$\xi = a_0 + a_1.x^c + a_2.y^c + a_3.x^c.y^c$$
$$\eta = b_0 + b_1.x^c + b_2.y^c + b_3.x^c.y^c \quad (1.48)$$

1.5. Outer orientation

Outer orientation is based on usage of co-linearity conditions. To determine the parameters of outer orientation of photo at least 3 points with given coordinates are necessary.

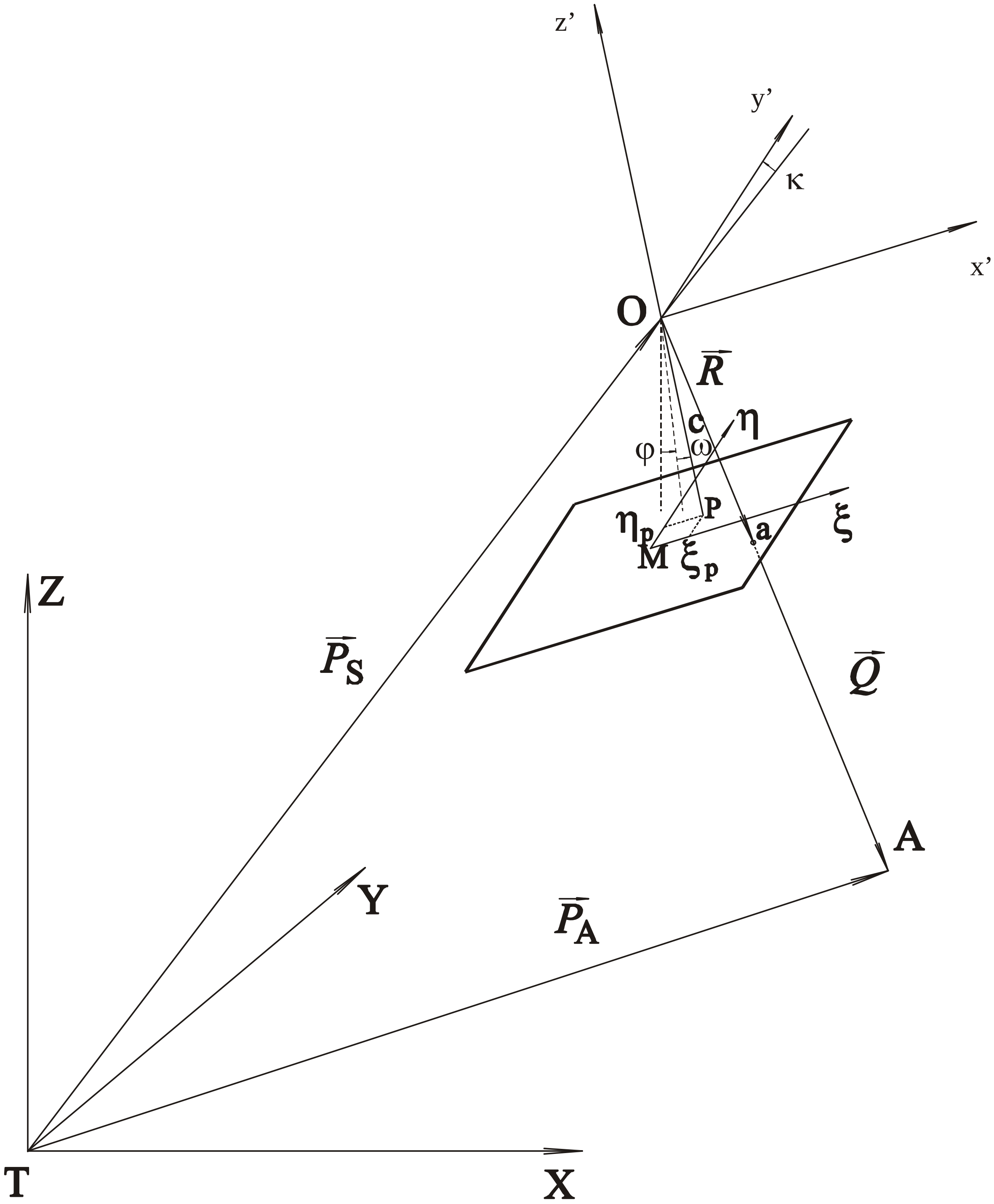

Figure 1.12. Outer orientation

The elements of outer orientation of photo are six – three linear and three angular.

The linear elements are co-ordinates of the projection center (X_0, Y0, Z0) and the angular ones are angles of rotation of coordinate system ω, φ, κ.

The co-linearity equations are used for determination of elements of outer orientation

$$
\begin{aligned}
\xi - \xi_p &= -c.\frac{r_{11}.(X_i - X_0) + r_{21}.(Y_i - Y_0) + r_{31}.(Z_i - Z_0)}{r_{13}.(X_i - X_0) + r_{23}.(Y_i - Y_0) + r_{33}.(Z_i - Z_0)} \\
\eta - \eta_p &= -c.\frac{r_{12}.(X_i - X_0) + r_{22}.(Y_i - Y_0) + r_{32}.(Z_i - Z_0)}{r_{13}.(X_i - X_0) + r_{23}.(Y_i - Y_0) + r_{33}.(Z_i - Z_0)}
\end{aligned}
\qquad (1.49)
$$

The every point with known co-ordinates (control point) gives two equations, so 3 points with known coordinates are necessary. It is necessary to apply adjustment by least square method when the number of points exceeds three.

1.6.Usage of homogeneous coordinates in Photogrammetry

The application of homogeneous coordinates requires addition of extra n+1 coordinate for n-dimensional coordinate system. This allows not only rotation but also translation and scaling to be represented by common rotational matrix with size (n+1)x(n+1). For 3D space this leads to four-dimensional vectors usage.

$$
r = \mathbf{T}.\mathbf{R} \qquad (1.50)
$$

$$
\mathbf{T} = \begin{bmatrix} \mathbf{t_{11}} & \mathbf{t}_{12} & \mathbf{t}_{13} & \mathbf{t}_{14} \\ \mathbf{t_{21}} & \mathbf{t}_{22} & \mathbf{t}_{23} & \mathbf{t}_{24} \\ \mathbf{t_{31}} & \mathbf{t}_{32} & \mathbf{t}_{33} & \mathbf{t}_{34} \\ \mathbf{t_{41}} & \mathbf{t}_{42} & \mathbf{t}_{43} & \mathbf{t}_{44} \end{bmatrix} \quad R = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \quad r = \begin{bmatrix} x \\ y \\ z \\ h \end{bmatrix}
$$

where T is transformation matrix,
R – object space vector,
r – image space vector (mapping space).

If h is used for normalising of r vector, then h is scale factor. In respect to this the last column and last row coefficients have the following functions:

$\left\{t_{i4}, i = 1:3\right\}$ - translation in 3D space;

$\left\{t_{4j}, j = 1:3\right\}$ projective transformation;

t_{44} - common scaling,

$\left\{t_{jj}, j = 1:3\right\}$ coordinate axes scaling.

The normalization of vector is described by the relation:

$$\begin{bmatrix} x^n \\ y^n \\ z^n \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{x}{h} \\ \frac{y}{h} \\ \frac{z}{h} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{x}{t_{41}X + t_{42}Y + t_{43}Z + t_{44}} \\ \frac{y}{t_{41}X + t_{42}Y + t_{43}Z + t_{44}} \\ \frac{z}{t_{41}X + t_{42}Y + t_{43}Z + t_{44}} \\ 1 \end{bmatrix} \quad (1.51)$$

The matrix of homogeneous coordinates could be divided into four sub-matrixes, respectively for scaling, affine transformation, translation and projective transformation.

$$\mathbf{T} = \mathbf{T}_S . \mathbf{T}_A . \mathbf{T}_M . \mathbf{T}_P \ , \quad (1.52)$$

where matrixes T_S, T_A , T_M and T_P have the presentation:

$$\mathbf{T}_S = \begin{bmatrix} t_{11}^S & 0 & 0 & 0 \\ 0 & t_{22}^S & 0 & 0 \\ 0 & 0 & t_{33}^S & 0 \\ 0 & 0 & 0 & t_{44}^S \end{bmatrix} \text{ scaling,} \quad (1.53)$$

$$\mathbf{T}_A = \begin{bmatrix} t_{11}^A & t_{12}^A & t_{13}^A & 0 \\ t_{21}^A & t_{22}^A & t_{23}^A & 0 \\ t_{31}^A & t_{32}^A & t_{33}^A & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ affine transformation,} \quad (1.54)$$

$$\mathbf{T}_M = \begin{bmatrix} 1 & 0 & 0 & t_{14}^M \\ 0 & 1 & 0 & t_{24}^M \\ 0 & 0 & 1 & t_{34}^M \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ translation,} \quad (1.55)$$

$$\mathbf{T}_P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_{41}^P & t_{42}^P & t_{43}^P & 1 \end{bmatrix} \text{ projective transformation.} \quad (1.56)$$

In case when columns of matrix T_A are ortho vectors, (vectors, for which the scalar multiplication is 0), then matrix describes only rotation in space.

According to the matrix presentation and the order of multiplication the coefficients of the whole matrix are:

$$\mathbf{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} = \begin{bmatrix} t_{11}^{S} t_{11}^{A} & t_{11}^{S} t_{12}^{A} & t_{11}^{S} t_{13}^{A} & t_{11}^{S} t_{11}^{M} \\ t_{22}^{S} t_{21}^{A} & t_{22}^{S} t_{22}^{A} & t_{22}^{S} t_{23}^{A} & t_{22}^{S} t_{24}^{M} \\ t_{33}^{S} t_{31}^{A} & t_{33}^{S} t_{32}^{A} & t_{33}^{S} t_{33}^{A} & t_{33}^{S} t_{34}^{M} \\ t_{44}^{S} t_{41}^{P} & t_{44}^{S} t_{42}^{P} & t_{44}^{S} t_{43}^{P} & t_{44}^{S} \end{bmatrix} . \qquad (1.57)$$

There are of interest some special types of transformations, which are corresponding to the projective transformation. For projective transformation over the plane z=0 from point, lying over the z axes, the matrix has the presentation:

$$\mathbf{T}_{\mathrm{Pz}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & f & 1 \end{bmatrix} . \qquad (1.58)$$

Projective transformation over the plane z=0 from arbitrary point in space is described by the matrix:

$$\mathbf{T}_{\mathrm{Pz}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & d & f & 1 \end{bmatrix} . \qquad (1.59)$$

From the photogrammetric point of view the coefficients c and d are principle point coordinates (x_p, y_p), and f is camera constant c. In this case the relation for aerial photos are described by the matrix relation:

$$\begin{bmatrix} x \\ y \\ 0 \\ h \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ 0 & 0 & 0 & 0 \\ c & d & f & m \end{bmatrix} . \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} . \qquad (1.60)$$

This relation can be used for orientation or restitution of space coordinates from two photos.