UNIVERSITY OF ARCHITECTURE, CIVIL ENGINEERING AND GEODESY

TECHNICAL MECHANICS DEPARTMENT

S O L V E D E X A M P L E S

OF

C O U R S E W O R K S

ON

MECHANICS – PART I

(KINEMATICS & STATICS)

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Sofia

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COURSE WORK 1: KINEMATICS OF THE PLANE MOTION OF RIGID BODIES

Problem 1.1

At the instant given, a mechanism occupies position shown in Fig.1.1.1. Velocity and acceleration of point A are $V_A = 15$ m/s and $a_A = 36$ m/s^2, respectively. Cylinder rolls without sliding.

Determine:

1. Velocities of points B, C, D, E and angular velocities of all members of the mechanism;
2. Acceleration of point B and angular accelerations of members AB and OB.

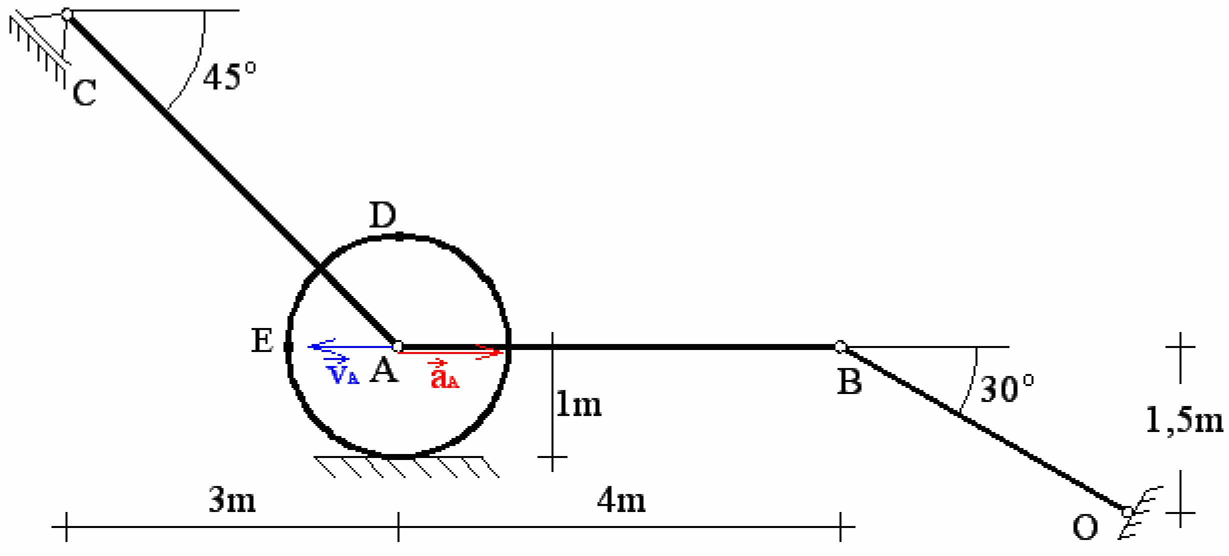


Fig. 1.1.1

Solution:

1. Numeration of the mechanism's members and determination of their types of motion

Members of the mechanism are four – cylinder of centre A (member 1), rod AB (member 2), rod AC (member 3) and rod OB (member 4). The types of their motion are found by consideration of previous instant of the motion of the mechanism, i.e. mechanism's position $O'B'A'C'$ (Fig.1.1.2).

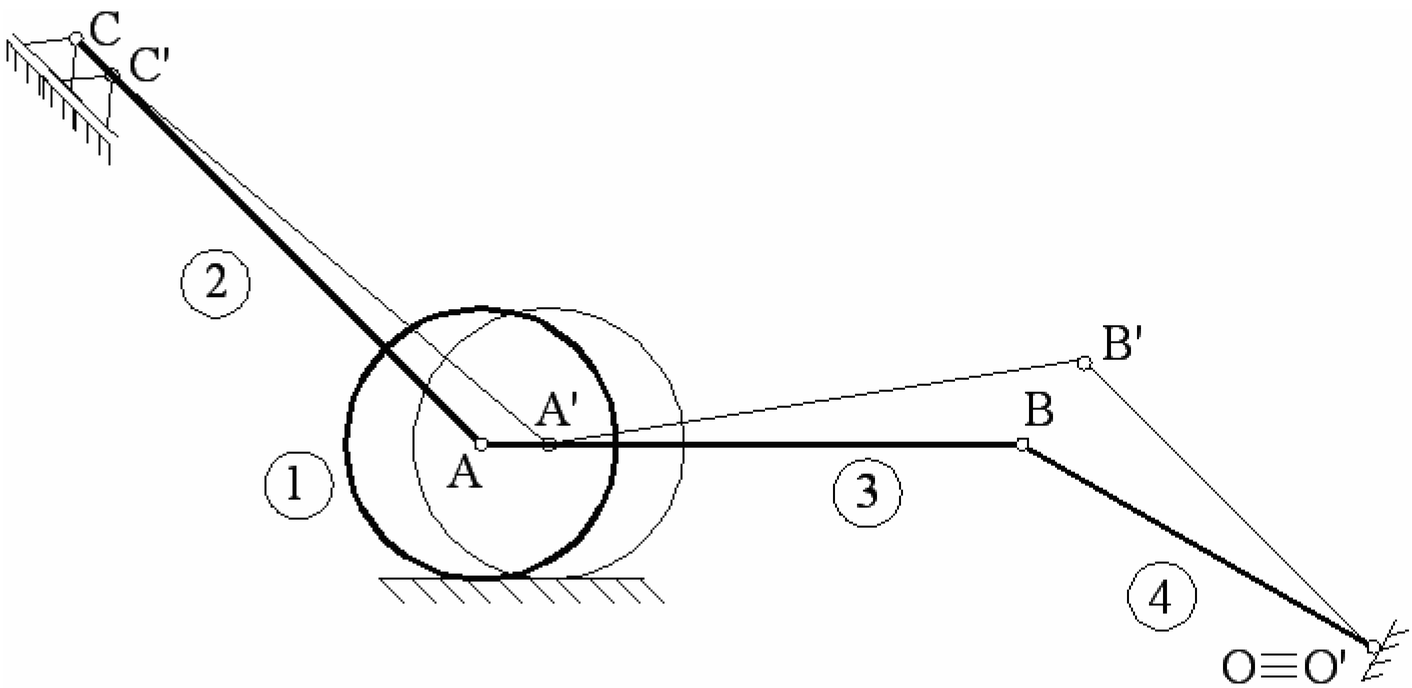


Fig. 1.1.2

Cylinder (1) rolls, i.e. it performs plane motion. Members (2) and (3) also perform plane motion because they have no fixed points and do not remain parallel to themselves during the motion. Member (4) has an unmovable point *O*, i.e. it performs rotation about point *O*.

2. Determination of velocities of points B, C, D, E and angular velocities of all members of the mechanism

The procedure of determination of all velocities begins from the given velocity of point A which is a point of three mechanism's members, namely (1), (2) and (3).

Cylinder performs plane motion represented as a rotation about instantaneous center of zero velocity, the point at which cylinder touches the ground. Then, the magnitude of the angular velocity of cylinder (1) is:

$$\omega_1 = \frac{V_A}{AP_1} = \frac{15}{1} = 15\,\text{s}^{-1},$$

where $\overline{AP_1}$ is the distance from point A to the cylinder's instantaneous center of zero velocity. The sense of ω_1 follows the sense of $\vec{V}_A$ (Fig.1.1.3).

Further, the velocity of point D is obtained as:

$$V_D = \omega_1 . DP_1 = 15.2 = 30\,\text{m/s},$$

where $\overline{DP_1}$ is the distance between D and the cylinder's instantaneous center of zero velocity. $\vec{V}_D$ is perpendicular to $\overline{DP_1}$, and its sense depends on the sense of ω_1.

After that, the velocity of point E is found as:

$$V_E = \omega_1 . EP_1 = 15.\sqrt{2} = 21{,}21\,\text{m/s},$$

where $\overline{EP_1}$ is the distance between E and the cylinder's instantaneous center of zero velocity. $\vec{V}_E$ is perpendicular to $\overline{EP_1}$, and its sense follows the sense of ω_1.

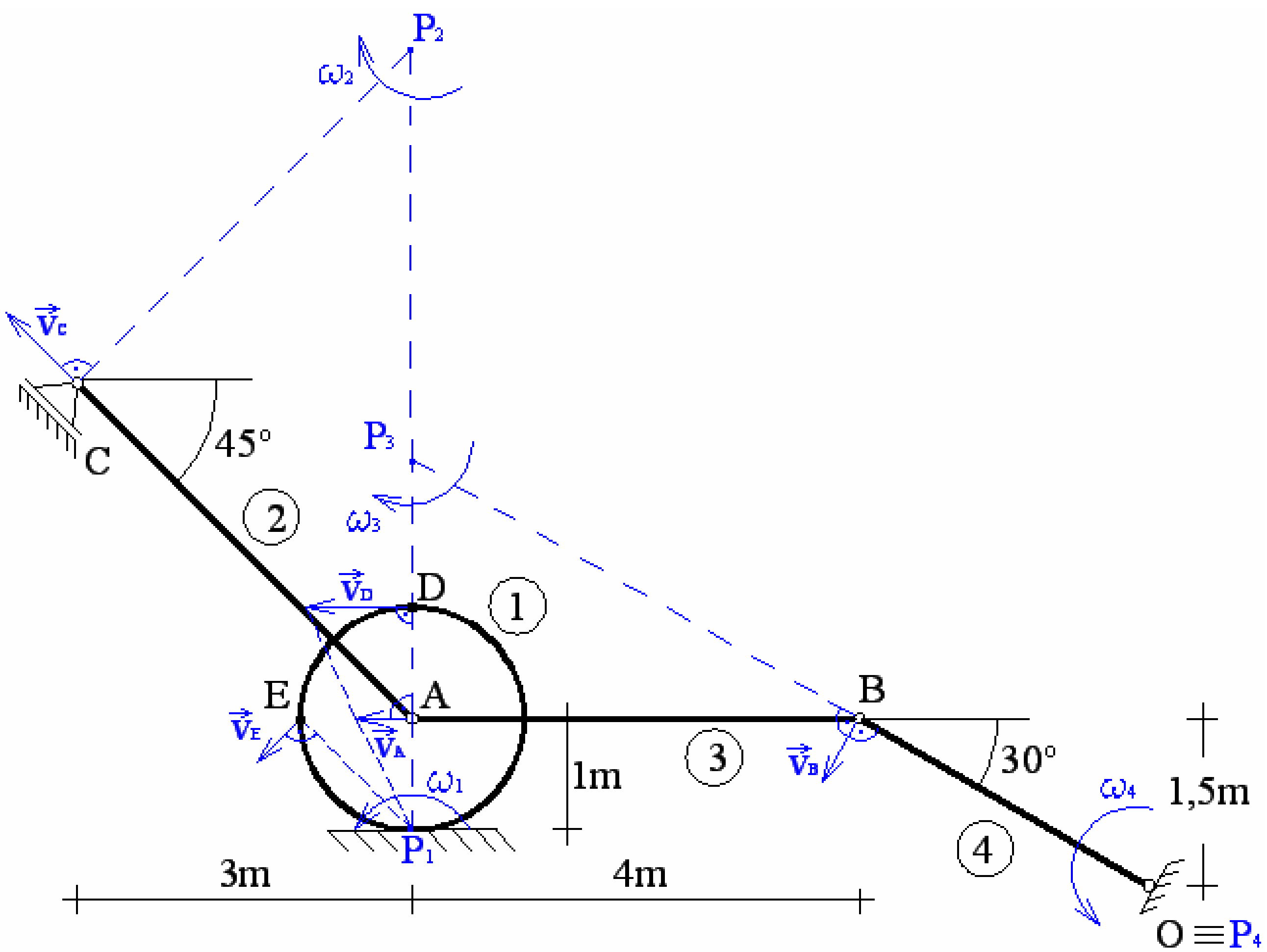


Fig. 1.1.3

Solution continues with determination of the angular velocity of member (2) performing plane motion. Such motion is presented as a rotation about the instantaneous center of zero velocity of body (2). Here, in order to find its position, perpendiculars to the directions of the velocities of two points are drawn. These points are A and C, because the directions of their velocities are known: direction of $\vec{V}_A$ is horizontal, while direction of $\vec{V}_C$ is inclined at an angle of 45^0 with respect to the horizontal. Then, intersecting the perpendiculars, the instantaneous center of zero velocity, point P_2, is located (Fig.1.1.3). Hence:

$$\omega_2 = \frac{V_A}{AP_2},$$

where $\overline{AP_2}$ is the distance between A and P_2. To obtain $\overline{AP_2}$ the isosceles right-angled triangle ACP_2 is considered:

$$\frac{AC}{AP_2} = \cos 45^0 \Rightarrow AP_2 = \frac{AC}{\cos 45^0}; \quad \frac{3}{AC} = \cos 45^0 \Rightarrow AC = \frac{3}{\cos 45^0} = 4{,}24\,\text{m}; AP_2 = \frac{4{,}24}{\cos 45^0} = 6\,\text{m}.$$

Finally,

$$\omega_2 = \frac{15}{6} = 2{,}5\ \text{s}^{-1},$$

of sense following the sense of $\vec{V}_A$.

Further, the velocity of point C is determined using the expression:

$$V_C = \omega_2.CP_2,$$

where $CP_2 = AC = 4{,}24\,\text{m}$, because both are cathetus in the isosceles triangle ACP_2.

$$V_C = 2{,}5.4{,}24 = 10{,}6\,\text{m/s}.$$

$\vec{V}_C$ is perpendicular to $\overline{CP_2}$ and its sense is determined by the sense of ω_2 (Fig.1.1.3).

Next step is to determine the angular velocity of member (3). Again, the plane motion is presented as rotation and the velocity of point A is used. Location of the instantaneous center of zero velocity of member (3) is found by intersection of perpendiculars to the directions of velocities of points A and B (Fig.1.1.3). Then, the angular velocity of member (3) is:

$$\omega_3 = \frac{V_A}{AP_3},$$

where $\overline{AP_3}$ is obtained from the right-angled triangle ABP_3, as follows:

$$\frac{AP_3}{AB} = tg30^0 \Rightarrow AP_3 = AB.tg30^0 = 4.tg30^0 = 2{,}308\ \text{m}.$$

Thus,

$$\omega_3 = \frac{15}{2{,}308} = 6{,}5\ \text{s}^{-1},$$

where the sense depends on the sense of $\vec{V}_A$.

Further, velocity of point B is obtained applying the expression:

$$V_B = \omega_3.BP_3,$$

where $\overline{BP_3}$ is carried out by the Pythagoras's theorem as:

$$(BP_3)^2 = (AB)^2 + (AP_3)^2 \Rightarrow BP_3 = \sqrt{(AP_3)^2 + (AB)^2} = \sqrt{2{,}308^2 + 4^2} = 4{,}618\,\text{m}.$$

Hence,

$$V_B = 6{,}5.4{,}618 = 30\ \text{m/s},$$

of sense following the sense of ω_3.

Finally, $\vec{V}_B$ is used for obtaining of the angular velocity of member (4):

$$\omega_4 = \frac{V_B}{OB} = \frac{30}{3} = 10\ \text{s}^{-1},$$

where the distance $\overline{OB}$ is carried out as:

$$\frac{1{,}5}{OB} = \sin 30^0 \;\Rightarrow\; OB = \frac{1{,}5}{\sin 30^0} = 3 \text{ m.}$$

- **Check with the theorem of the velocities projections:**

$\text{pr.}_{AE}\vec{V}_A = \text{pr.}_{AE}\vec{V}_E \;\Rightarrow\; V_A \cos 0^0 = V_E \cos 45^0 \;\Rightarrow\; 15.\cos 0^0 = 21{,}21.\cos 45^0 \;\Rightarrow\; 15 = 15;$

$\text{pr.}_{DE}\vec{V}_D = \text{pr.}_{DE}\vec{V}_E \;\Rightarrow\; V_D \cos 45^0 = V_E \cos 0^0 \;\Rightarrow\; 30.\cos 45^0 = 21{,}21.\cos 0^0 \;\Rightarrow\; 21{,}21 = 21{,}21;$

$\text{pr.}_{AB}\vec{V}_A = \text{pr.}_{AB}\vec{V}_B \;\Rightarrow\; V_A \cos 0^0 = V_B \cos 60^0 \;\Rightarrow\; 15.\cos 0^0 = 30.\cos 60^0 \;\Rightarrow\; 15 = 15;$

$\text{pr.}_{AC}\vec{V}_A = \text{pr.}_{AC}\vec{V}_C \Rightarrow V_A \cos 45^0 = V_C \cos 0^0 \Rightarrow 15.\cos 45^0 = 10{,}6.\cos 0^0 \Rightarrow 10{,}6 = 10{,}6;$

3. Determination of acceleration of point B and angular accelerations of AB and OB

Point B is the connecting point of members (3) and (4). It should be noted that the magnitude and direction of acceleration of point B are unknown. Therefore, in order to find them, the following procedure is performed. First, B is considered as a point of member (3). Thus, the formula of the acceleration is:

$$\vec{a}_B = \vec{a}_A + \vec{a}^c_{BA} + \vec{a}^r_{BA}, \qquad \text{(I)}$$

where $\vec{a}_A$ is the acceleration of point A chosen for a pole, $\vec{a}^c_{BA}$ and $\vec{a}^r_{BA}$ are the centripetal and rotational components, respectively, of the acceleration of point B. There are three unknowns in the equation (I) – the magnitude and sense of $\vec{a}_B$ and the magnitude of $\vec{a}^r_{BA}$, i.e. the problem is insoluble.

Further, B is examined as a point of member (4). Then, the expression of the acceleration is:

$$\vec{a}_B = \vec{a}^n_B + \vec{a}^t_B, \qquad \text{(II)}$$

where $\vec{a}^n_B$ and $\vec{a}^t_B$ are the normal and tangential components, respectively. Three unknowns are available in equation (II) – the magnitude and sense of $\vec{a}_B$ and the magnitude of $\vec{a}^t_B$. It means that the problem is insoluble again. However, equating the right-hand sides of (I) and (II), it is obtained:

$$\vec{a}^n_B + \vec{a}^t_B = \vec{a}_A + \vec{a}^c_{BA} + \vec{a}^r_{BA}. \qquad \text{(III)}$$

Thus, two unknowns take part in expression (III) – the magnitudes of $\vec{a}^r_{BA}$ and $\vec{a}^t_B$. Then, the projections of (III) onto two axes make the task soluble.

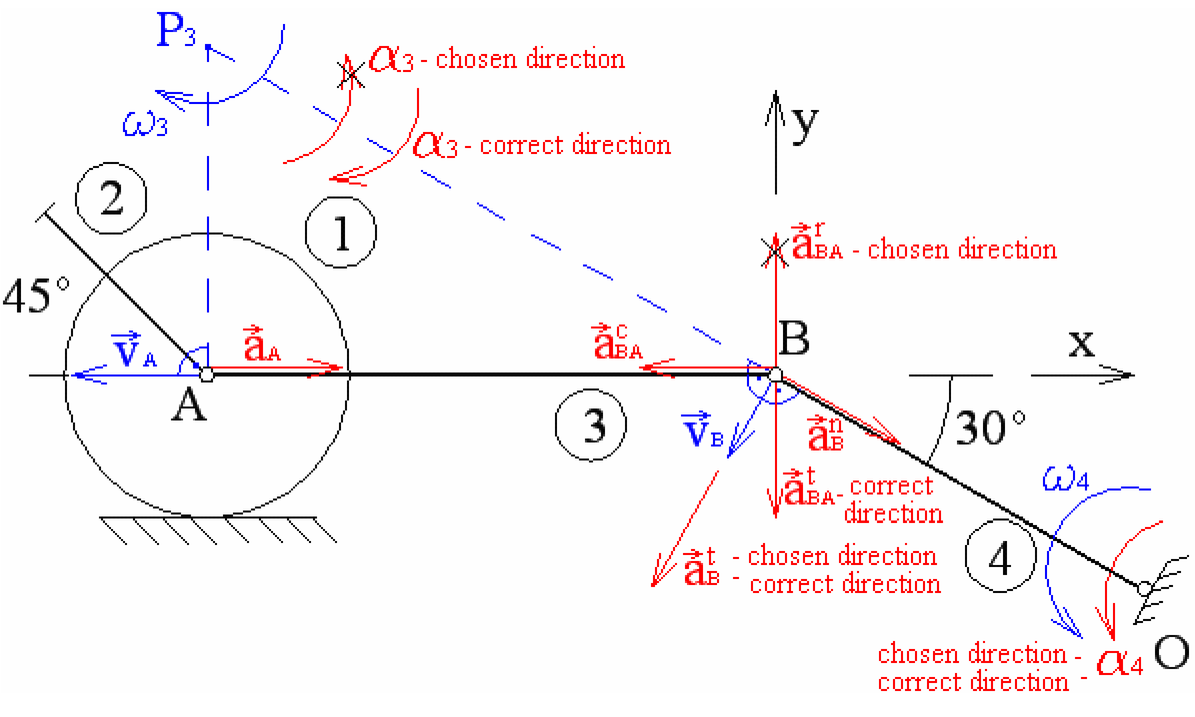


Fig. 1.1.4

First, the normal and centripetal accelerations in (III) are carried out as:

$$a^n_B = \omega_4^2.OB = 10^2.3 = 300 \text{ m/s}^2, \qquad a^c_{BA} = \omega_3^2.BA = 6{,}5^2.4 = 169 \text{ m/s}^2,$$

where the sense of a^n_B is from B to O, while the sense of a^c_{BA} is from B to A (Fig.1.1.4).

Further, the tangential and rotational accelerations are introduced perpendicular to the normal and centripetal ones, respectively. Their magnitudes are:

$$a_B^t = \alpha_4.OB = 3\alpha_4, \qquad a_{BA}^r = \alpha_3.BA = 4\alpha_3,$$

where α_3 and α_4 are the angular accelerations of the members (3) and (4), respectively. The senses of $\vec{a}_{BA}^r$, $\vec{a}_B^t$, α_3, and α_4 are arbitrary chosen and shown in Fig.1.1.4.

Then, the projection of (III) onto x-axis is written, as follows:

$$a_B^n \cos 30^0 - a_B^t \sin 30^0 = a_A - a_{BA}^c,$$

$$300.0,866 - a_B^t.0,5 = 36 - 169,$$

$$a_B^t = \frac{259,81 - 36 + 169}{0,5} = 785,62 \text{ m/s}^2,$$

$$\alpha_4 = \frac{a_B^t}{3} = \frac{785,62}{3} = 261,87 \text{ s}^{-2}.$$

$\vec{a}_B^t$ and α_4 have positive signs, i.e. their senses are correctly chosen (Fig.1.1.4).

Further, the projection of (III) onto y-axis is written as:

$$-a_B^n \sin 30^0 - a_B^t \cos 30^0 = a_{BA}^r,$$

$$-300.0,5 - 785,62.0,866 = a_{BA}^r,$$

$$a_{BA}^r = -150 - 680,35 = -830,35 \text{ m/s}^2,$$

$$\alpha_3 = \frac{a_{BA}^r}{4} = \frac{-830,35}{4} = -207,59 \text{ s}^{-2}.$$

The magnitudes of $\vec{a}_{BA}^r$ and α_3 are obtained negative meaning that their senses are incorrect and have to be changed. (Fig.1.1.4).

Finally, in order to find the total acceleration of point B, equation (II) is used, because the components $\vec{a}_B^n$ and $\vec{a}_B^t$ are perpendicular to each other. Then:

$$a_B = \sqrt{(a_B^ц)^2 + (a_B^{вр})^2} = \sqrt{300^2 + 785,62^2} = 840,95 \text{ m/s}^2.$$

The sense and direction of $\vec{a}_B$ are determined using parallelogram rule (Fig.1.1.5).

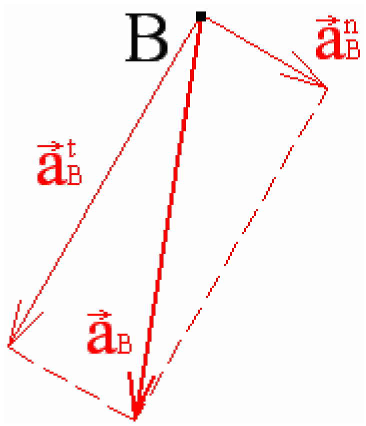


Fig. 1.1.5

COURSE WORK 2: KINEMATICS OF THE RELATIVE MOTION OF A POINT

Problem 2.1

A pipe is pivoted at one end to point O and starts to rotate from horizontal position according to equation $\varphi(t) = \dfrac{t^2}{3}$ [φ – rad, t – s] (Fig.2.1.1). Meanwhile, point M is moving relative to the pipe in accordance with equation $S(t) = \overline{OM}(t) = t^2 + 0{,}5t$ [S – m, t – s]. Find the absolute velocity $\vec{V}$ and the absolute acceleration $\vec{a}$ of the point M at instant $t_1 = 2$ s.

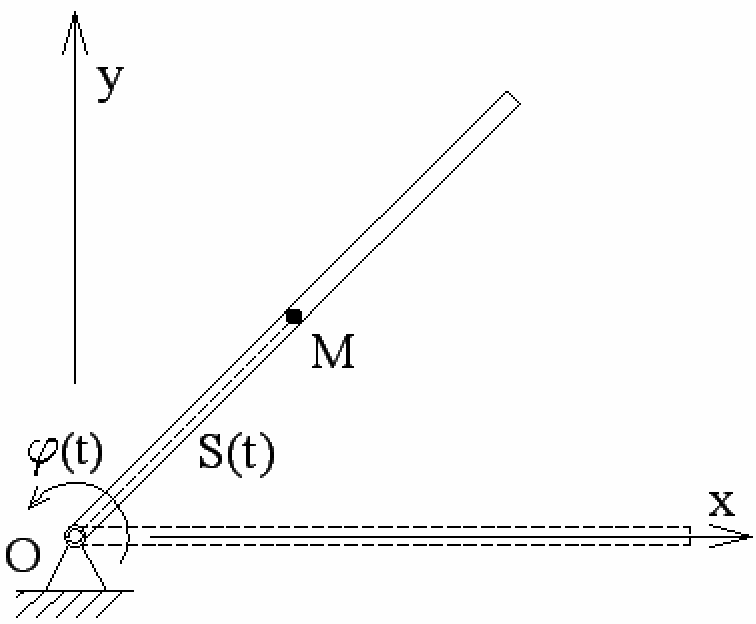


Fig. 2.1.1

Solution:

Rotation of the pipe in accordance with equation $\varphi(t) = \dfrac{t^2}{3}$ is the motion that transports point M. The rectilinear motion of the point according to equation $S(t) = \overline{OM}(t) = t^2 + \dfrac{t}{2}$ is the relative motion.

1. Position of the point at instant $t_1 = 2$ s

First step of the solution is determination of the point M's position at instant $t_1 = 2$ s. This is performed by substitution of $t = t_1 = 2$ s into the equations of motions. The result is:

$$\varphi(t_1) = \frac{t_1^2}{3} = \frac{4}{3}\,\text{rad} = 76{,}4^0;$$

$$S(t_1) = \overline{OM}(t_1) = t_1^2 + 0{,}5t_1 = 2^2 + 0{,}5.2 = 5\,\text{m}.$$

2. Absolute velocity of the point at instant $t_1 = 2$ s

The absolute velocity of point M is equal to the sum of relative and transport velocity of the point:

$$\vec{V} = \vec{V}_r + \vec{V}_e.$$

The relative velocity is the first order derivative with respect to the time of the relative motion equation:

$$V_r(t) = \dot{S}(t) = 2t + 0{,}5 \;\Rightarrow\; V_r(t_1) = 2.2 + 0{,}5 = 4{,}5\,\text{m/s}.$$

$\vec{V}_r$ is obtained positive meaning that its sense coincides with the sense of the relative motion (Fig.2.1.2).

The pipe performs rotation. Then, the transport velocity of the point is obtained using the expression:

$$V_e = \omega.OM,$$

where ω is the angular velocity of the pipe, and OM is the distance from the center of rotation to the point.

The angular velocity is the first order derivative with respect to the time of the equation of rotation:

$$\omega(t) = \dot{\varphi}(t) = \frac{2t}{3} \Rightarrow \omega(t_1) = \frac{2t_1}{3} = \frac{2.2}{3} = 1{,}333\,\text{s}^{-1}.$$

It has a positive magnitude and its sense follows the sense of $\varphi(t)$ (Fig.2.1.2).

Finally, the magnitude of transport velocity of M is:

$$V_e = 1{,}333.5 = 6{,}665\,\text{m/s},$$

The $\vec{V}_e$'s direction is perpendicular to OM (trajectory of the point due to the transport motion is circle meaning that its direction is perpendicular to the radius) with the sense depending on the sense of ω (Fig.2.1.2).

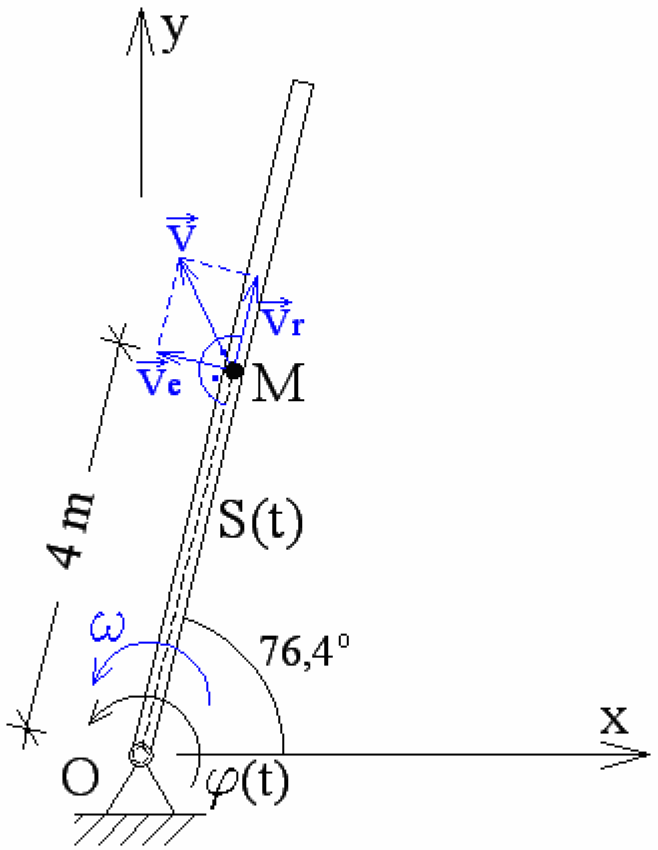


Fig. 2.1.2

Further, since the relative and transport velocity of the point are perpendicular to each other (Fig.2.1.2), the absolute velocity is obtained directly by the following expression:

$$V = \sqrt{V_r^2 + V_e^2} = \sqrt{4{,}5^2 + 6{,}665^2} = 8{,}04\,\text{m/s}.$$

The sense and direction of the absolute velocity are found by the parallelogram law (Fig.2.1.2).

3. Absolute acceleration of the point at instant $t_1 = 2\ s$

The absolute acceleration of the point M is equal to the sum of the relative, transport and Coriolis accelerations:

$$\vec{a} = \vec{a}_r + \vec{a}_e + \vec{a}_C.$$

The relative motion is rectilinear. Then, the relative acceleration is the second order derivative with respect to the time of the relative motion equation:

$$a_r(t) = \ddot{S}(t) = 2 \Rightarrow a_r(t_1) = 2\,\text{m/s}^2.$$

It is obtained positive, i.e. its sense coincides with the sense of the point relative motion (Fig.2.1.3).

Since the transport motion of the point is rotation, the transport acceleration has two components – normal and tangential:

$$\vec{a}_e = \vec{a}_e^n + \vec{a}_e^t.$$

The normal acceleration is:

$$a_e^n = \omega^2.OM = 1{,}333^2.5 = 8{,}884\ \text{m/s}^2,$$

and it points to the center of rotation, here, point O (Fig.2.1.3).

The tangential component is carried out by the expression:

$$a_e^t = \alpha.OM \ ,$$

where α is the angular acceleration of the pipe, which is the second order derivative with respect to the time of the equation of rotation:

$$\alpha(t) = \ddot{\varphi}(t) = \frac{2}{3} \ \Rightarrow \ \alpha(t_1) = \frac{2}{3} = 0{,}667 \text{ s}^{-2},$$

It has positive magnitude, i.e. its sense follows the sense of $\varphi(t)$ (Fig.2.1.3). Then:

$$a_e^t = 0{,}667.5 = 3{,}335 \ \text{ m/s}^2,$$

which is perpendicular to OM , and of sense determined by the sense of α .

The magnitude of the transport acceleration is:

$$a_e = \sqrt{(a_e^n)^2 + (a_e^t)^2} = \sqrt{8{,}884^2 + 3{,}335^2} = 9{,}489 \ \text{ m/s}^2.$$

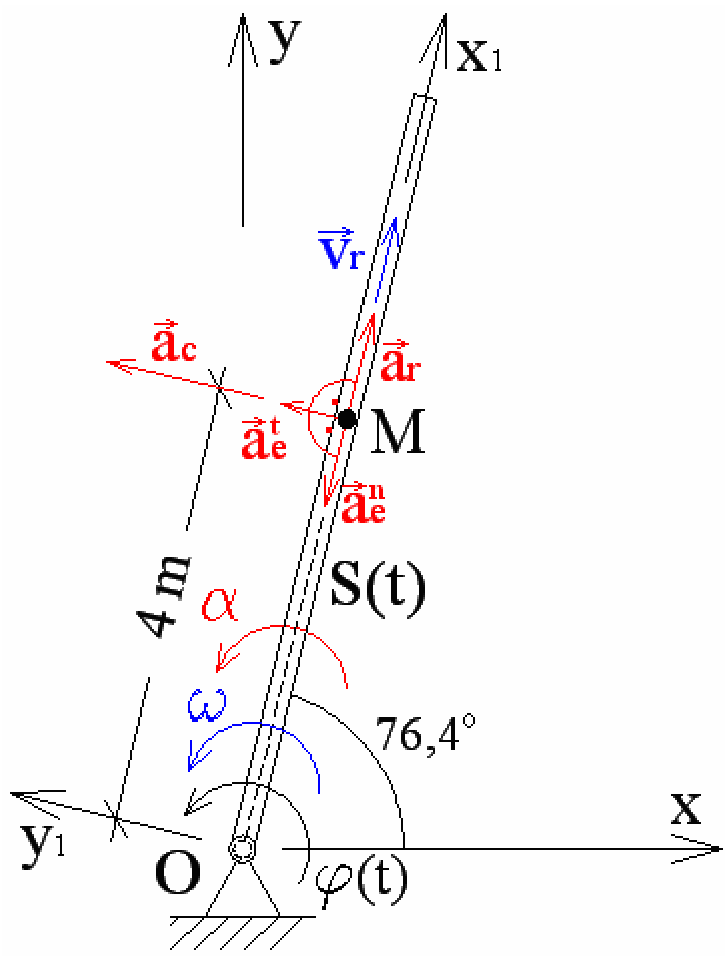


Fig. 2.1.3

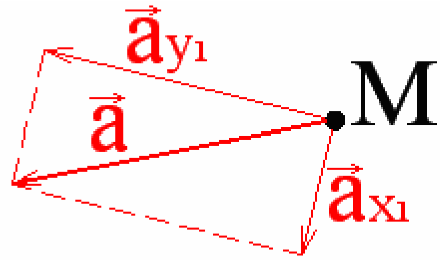


Fig. 2.1.4

The Coriolis acceleration is caused by the interaction between the angular velocity of the pipe and the relative velocity of the point. It has to be obtained by the following formula:

$$\vec{a}_C = 2\vec{\omega} \times \vec{V}_r \ .$$

The magnitude of the Coriolis acceleration is:

$$a_C = 2\omega.V_r.\sin(\vec{\omega};\vec{V}_r) \ .$$

Here, the angle between the directions of angular and relative velocity is 90^0, because $\vec{\omega}$ is perpendicular to the plane xOy (its direction is determined by the right-hand rule – the four fingers of the right hand follows the direction of rotation when the thumb points in the sense of the vector), while $\vec{V}_r$ lies in the plane xOy .

Thus:

$$a_C = 2.1{,}33.4{,}5.\sin 90^0 = 2.1{,}333.4{,}5.1 = 12 \ \text{ m/s}^2.$$

According to the cross product, vector $\vec{a}_C$ has to be perpendicular to the plane formed by $\vec{\omega}$ and $\vec{V}_r$, and, besides, the three vectors have to form the right-handed system. Therefore, the direction and the sense of the Coriolis acceleration coincides with the direction of the transport tangential acceleration of the point (Fig.2.1.3).

Finally, the magnitude of the absolute acceleration of the point is obtained by projections of its components onto axes x_1 and y_1, which are perpendicular to each other. These axes are chosen because all of the accelerations can be projected with their total magnitudes onto them.

$$a = \sqrt{a_{x1}^2 + a_{y1}^2},$$

$$a_{x1} = a_r - a_e^n = 2 - 8{,}884 = -6{,}884 \text{ m/s}^2,$$

$$a_{y1} = a_e^t + a_c = 3{,}335 + 12 = 15{,}335 \text{ m/s}^2,$$

$$a = \sqrt{(-6{,}884)^2 + 15{,}335^2} = 16{,}81 \text{ m/s}^2.$$

The sense and direction of $\vec{a}$ are determined by the parallelogram rule (Fig. 2.1.4).

Problem 2.2

A semicircle-shaped body is attached to *z*-axis and rotates according to equation $\varphi(t) = 3t - t^2$ [φ – rad, t – s] while a point M moves on its periphery according to equation $S(t) = AM(t) = \pi.t^2$ [S – m, t – s]. Find the absolute velocity $\vec{V}$ and the absolute acceleration $\vec{a}$ of the point at instant $t_1 = 1$ s, if at the same instant the body occupies the position shown in Fig.2.2.1.

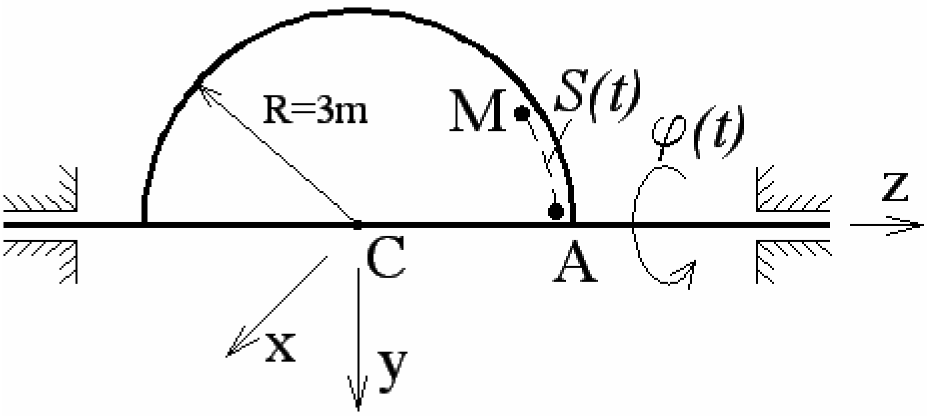


Fig. 2.2.1

Solution:

The motion that transports the point is rotation of the body in accordance with equation $\varphi(t) = 3t - t^2$. The relative motion is the motion of the point with respect to the body in accordance with equation $S(t) = AM(t) = \pi.t^2$. This motion is curvilinear, because the trajectory of the point is circle of radius *R*.

1. Position of the point at instant $t_1 = 1$ s

The task is to find the absolute velocity and acceleration $\vec{a}$ of point M at instant $t_1 = 1$ s. However, the position of the semicircle-shaped body at the same instant has been indicated by the statement. Therefore, only the position of point M relative to the body has to be found. Then, substituting $t = t_1 = 1$ s into the equation of the relative motion, it is obtained:

$$AM = S(1s) = \pi.1^2 = \pi \text{ [m]}.$$

Here, the more convenient way to represent the point position is using the central angle concept, i.e.:

$$\angle ACM = \frac{AM}{R} = \frac{\pi}{3} \text{[rad]} = \frac{\pi}{3}.\frac{180^0}{\pi} = 60^0.$$

2. Absolute velocity of the point at instant $t_1 = 1$ s

The absolute velocity of point M is equal to the sum of relative and transport velocity:

$$\vec{V} = \vec{V}_r + \vec{V}_e$$

The relative velocity is the first order derivative with respect to the time of the relative motion equation:

$$V_r(t) = \dot{S}(t) = 2\pi.t \quad \Rightarrow \quad V_r(t_1) = 2\pi.1 = 6{,}28 \text{ m/s}.$$

$\vec{V}_r$ lies on the circle's tangent at point M and its sense coincides with the sense of the relative motion (Fig.2.2.2).

Due to the rotation of the body, the transport velocity is obtain by the expression:

$$V_e = \omega.MN,$$

where ω is the angular velocity of the semicircle, and MN is the shortest distance between the point and the axis of rotation.

The angular velocity is the first order derivative with respect to the time of the equation of rotation:

$$\omega(t) = \dot{\varphi}(t) = 3 - 2t \quad \Rightarrow \quad \omega(1s) = 3 - 2.1 = 1 \text{ s}^{-1}.$$

It has a positive magnitude meaning that the sense coincides with the sense of $\varphi(t)$ (Fig.2.2.2).

Further, to find MN, the right-angled triangle MCN is considered:

$$\sin 60^0 = \frac{MN}{CM} \Rightarrow MN = CM.\sin 60^0 = R.\sin 60^0 = 3.0{,}866 = 2{,}6 \text{ m}.$$

Finally:

$$V_e = 1.2{,}6 = 2{,}6 \text{ m/s}.$$

The $\vec{V}_e$'s direction is perpendicular to MN, parallel to x-axis (the trajectory due to the rotation is circle and the velocity's direction is perpendicular to the radius) and the sense follows the sense of ω (Fig.2.2.2).

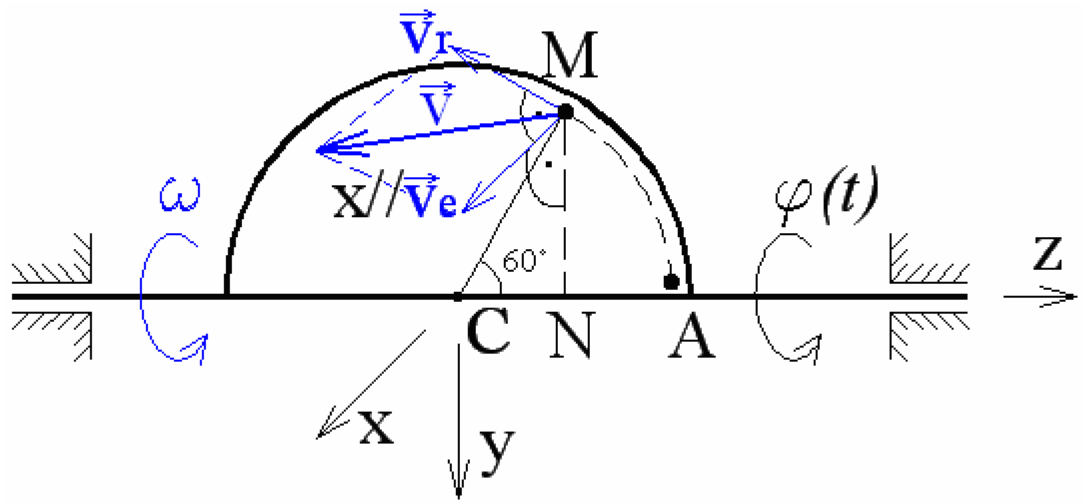


Fig. 2.2.2

Here, relative and transport velocity of the point have mutually perpendicular directions. Then, the magnitude of the velocity of the point is:

$$V = \sqrt{V_r^2 + V_e^2} = \sqrt{6{,}28^2 + 2{,}6^2} = 6{,}8 \text{ m/s}.$$

The sense and direction of the absolute velocity are determined by the parallelogram rule (Fig.2.2.2).

3. Absolute acceleration of the point at instant $t_1 = 1$ s

The absolute acceleration of the point M is equal to the sum of relative, transport and Coriolis components:

$$\vec{a}_a = \vec{a}_r + \vec{a}_e + \vec{a}_C.$$

Relative motion of the point is curvilinear and the relative acceleration has normal and tangential components:

$$\vec{a}_r = \vec{a}_r^n + \vec{a}_r^t.$$

When the point trajectory is a curve, the normal acceleration points to the center of such curve, and its magnitude is:

$$a_r^n = \frac{V_r^2}{R} = \frac{6{,}28^2}{3} = 13{,}15 \text{ m/s}^2.$$

The tangential component of relative acceleration is the second order derivative with respect to the time of the relative motion equation:

$$a_r^t(t) = \ddot{S}(t) = 2\pi \;\Rightarrow\; a_r^t(1s) = 6{,}28\ \text{m/s}^2.$$

$\vec{a}_r^t$ lies on the tangent to the trajectory of point M and by virtue of positive magnitude its sense coincides with the sense of the relative motion of the point (Fig.2.2.3).

The magnitude of the relative acceleration of M is:

$$a_r = \sqrt{(a_r^n)^2 + (a_r^t)^2} = \sqrt{13{,}15^2 + 6{,}28^2} = 14{,}57\ \text{m/s}^2.$$

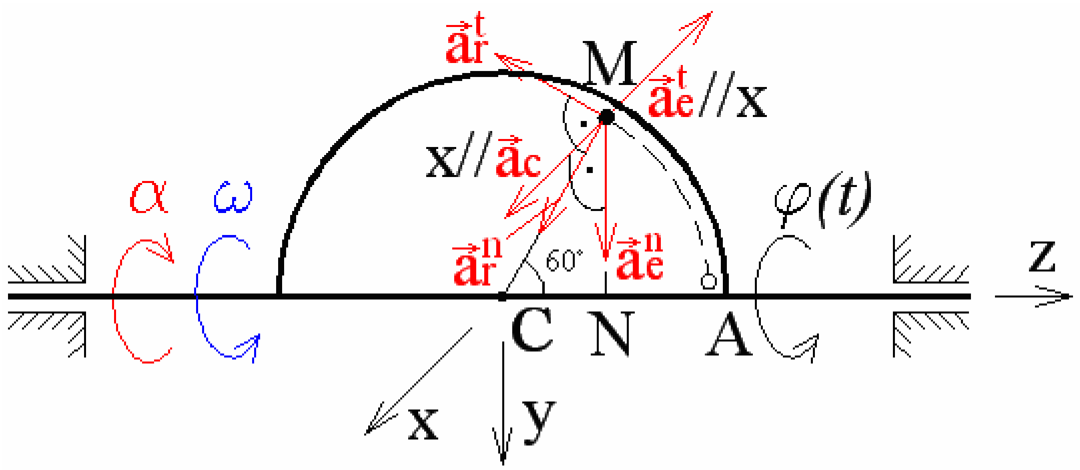


Fig. 2.2.3

Body performs rotation and the transport acceleration is the sum of normal and tangential accelerations:

$$\vec{a}_e = \vec{a}_e^n + \vec{a}_e^t.$$

The normal acceleration is given by the expression:

$$a_e^n = \omega^2.MN = 1^2.2{,}6 = 2{,}6\ \text{m/s}^2,$$

and it always directs to the axis of rotation (Fig.2.2.3).

The tangential component is:

$$a_e^t = \alpha.MN,$$

where α is the angular acceleration of the body and it is the second order derivative with respect to the time of the equation of rotation:

$$\alpha(t) = \ddot{\varphi}(t) = -2\ \text{s}^{-2} \;\Rightarrow\; \alpha(1\,\text{s}) = -2\ \text{s}^{-2}.$$

It is carried out with negative sign, i.e. its sense is opposite to the sense of $\varphi(t)$ (Fig.2.2.3).

$$a_e^t = 2.2{,}6 = 5{,}2\ \text{m/s}^2.$$

The sense of $\vec{a}_e^t$ coincides to this one of α and its direction is perpendicular to MN (Fig.2.2.3).

The magnitude of the transport acceleration of M is:

$$a_e = \sqrt{(a_e^{ц})^2 + (a_e^{вр})^2} = \sqrt{2{,}6^2 + 5{,}2^2} = 5{,}81\ \text{m/s}^2.$$

The Coriolis acceleration is:

$$\vec{a}_C = 2\vec{\omega} \times \vec{V}_r.$$

The magnitude of the Coriolis acceleration is given by the expression:

$$a_C = 2\omega.V_r.\sin(\vec{\omega};\vec{V}_r),$$

where the angle between the directions of angular and relative velocity is 150^0 (Fig.2.2.4).

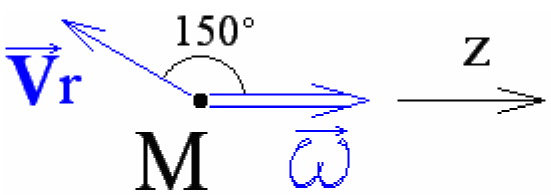


Fig. 2.2.4

Finally, ir is obtained:

$a_C = 2.1.6{,}28.\sin 150^0 = 6{,}28$ m/s^2.

Vector $\vec{a}_C$ has to be perpendicular to the plane formed by $\vec{\omega}$ and $\vec{V}_r$, and the three vectors must form the right-handed system. Therefore, the Coriolis acceleration has direction and sense as shown in Fig.2.2.5.

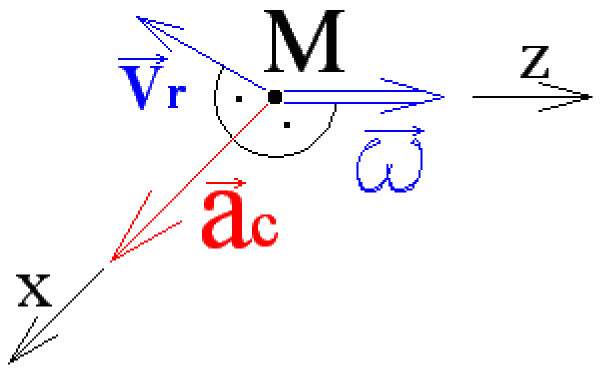


Fig. 2.2.5

To obtain the magnitude of the absolute acceleration of the point, the components have to be projected onto three mutually perpendicular axes. Here, these are the axes x, y and z. Then:

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}\ .$$

$a_x = -a_e^t + a_C = -5{,}2 + 6{,}28 = 1{,}08$ m/s^2;

$a_y = a_r^n \sin 60^0 - a_r^t \cos 60^0 + a_e^n$,

$a_y = 13{,}15.0{,}866 - 6{,}28.0{,}5 + 2{,}6 = 10{,}85$ m/s^2;

$a_z = -a_r^n \cos 60^0 - a_r^t \sin 60^0$,

$a_z = -13{,}15.0{,}5 - 6{,}28.0{,}866 = -12{,}01$ m/s^2;

$$a = \sqrt{1{,}08^2 + 10{,}85^2 + (-12{,}01)^2} = 16{,}22 \text{ m/s}^2$$

The sense and direction of $\vec{a}$ are determined by the parallelipiped rule (Fig. 2.2.6).

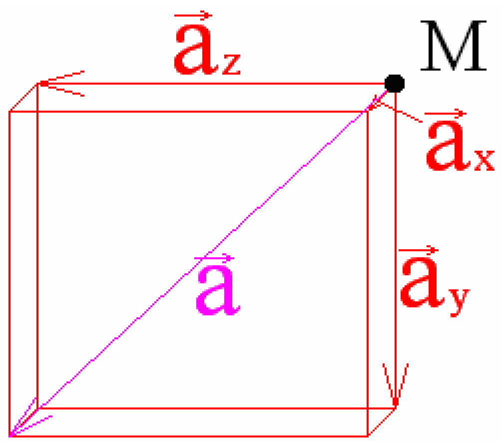


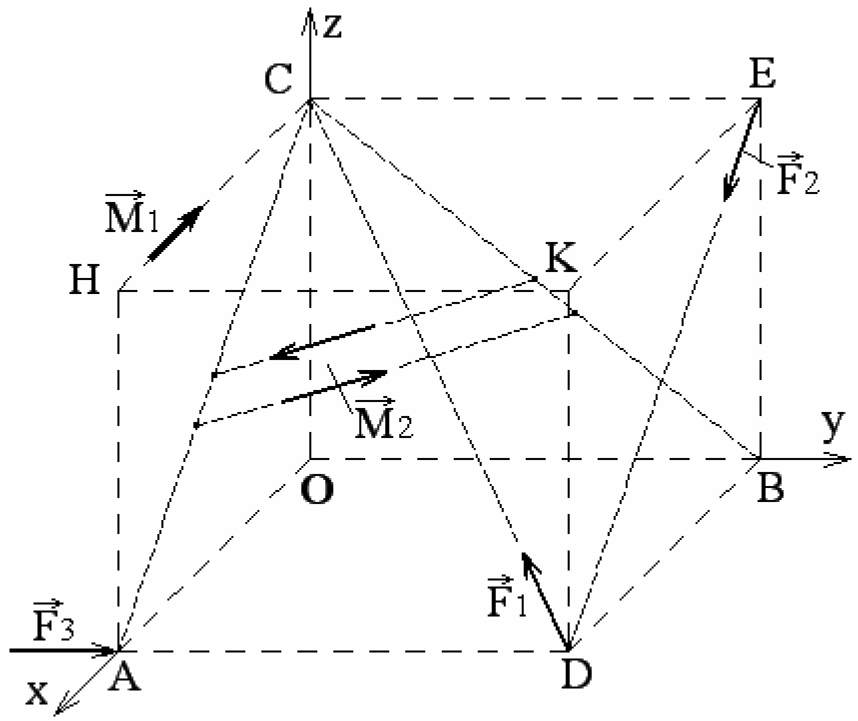
Fig. 2.2.6

COURSE WORK 3: REDUCTION OF A SPATIAL SET OF FORCES

Problem 3.1

A spatial set of forces is applied to homogeneous rectangular prism, as shown in Fig.3.1.1.

1. Reduce the set with respect to point O and determine the reduction case to which the set can be brought;
2. Draw on the sketch of suitable scale the force resultant, the moment resultant and the angle between them;
3. Calculate the moment of the force $\vec{F}_1$ about an axis formed by points E and H of direction from E to H.



Data:
$OA = 3\,\text{m}$
$OB = 5\,\text{m}$
$OC = 4\,\text{m}$

$F_1 = 56\,\text{kN}$
$F_2 = 32\,\text{kN}$
$F_3 = 28\,\text{kN}$
$M_1 = 25\,\text{kNm}$
$M_2 = 62\,\text{kNm}$

Fig. 3.1.1

Solution

Reduction of a set of forces about a point requires the force resultant and moment resultant about the point to be obtained.

1. Resultant force $\vec{R}$

The force resultant is going to be obtained adding its projections onto three mutually perpendicular axes, such as x, y, z:

$$\vec{R} = R_x\vec{i} + R_y\vec{j} + R_z\vec{k},$$

where:

$$R_x = F_{1x} + F_{2x} + F_{3x},$$
$$R_y = F_{1y} + F_{2y} + F_{3y},$$
$$R_z = F_{1z} + F_{2z} + F_{3z}.$$

Here, $F_{1x},\ldots,F_{3z}$ are the projections of the applied forces onto axes x, y, z, respectively. It should be noted that each one of them has to be taken with the sign depending on the sense of the force relative to the sense of the axes.

- Resolution of $\vec{F}_1$

Force $\vec{F}_1$ is in general position with respect to the axes x, y, and z. Therefore, to obtain its projections along the axes, the fact that $\vec{F}_1$ is collinear to vector $\overrightarrow{DC}$ has to be used. Thus:

$$F_{1x} = \lambda_{DC}F_1;\quad F_{1y} = \mu_{DC}F_1;\quad F_{1z} = \nu_{DC}F_1,$$

where λ_{DC}, μ_{DC}, ν_{DC} are the direction cosines of $\overrightarrow{DC}$. They are determined by the expressions:

$$\lambda_{DC} = \frac{x_C - x_D}{|DC|}; \qquad \mu_{DC} = \frac{y_C - y_D}{|DC|}; \qquad \nu_{DC} = \frac{z_C - z_D}{|DC|},$$

where $|DC| = \sqrt{(x_C - x_D)^2 + (y_C - y_D)^2 + (z_C - z_D)^2}$ is the magnitude of $\overrightarrow{DC}$:

$$|DC| = \sqrt{(0-3)^2 + (0-5)^2 + (4-0)^2} = 7{,}071\,\text{m}.$$

Further:

$$\lambda_{DC} = \frac{0-3}{7{,}071} = -0{,}4243; \qquad \mu_{DC} = \frac{0-5}{7{,}071} = -0{,}7071; \qquad \nu_{DC} = \frac{4-0}{7{,}071} = 0{,}5657.$$

- Check of the direction cosines:

$$\lambda_{DC}^2 + \mu_{DC}^2 + \nu_{DC}^2 = 1 \quad \Rightarrow \quad (-0{,}4243)^2 + (-0{,}7071)^2 + 0{,}5657^2 = 1 \quad \Rightarrow \quad 1 = 1\ !$$

Finally:

$$F_{1x} = -0{,}4243.56 = -23{,}76\,\text{kN}; \quad F_{1y} = -0{,}7071.56 = -39{,}6\,\text{kN}; \quad F_{1z} = 0{,}5657.56 = 31{,}68\,\text{kN}.$$

Negative signs show that F_{1x} and F_{1y} have senses opposite to the senses of the axes x and y (Fig.3.1.2).

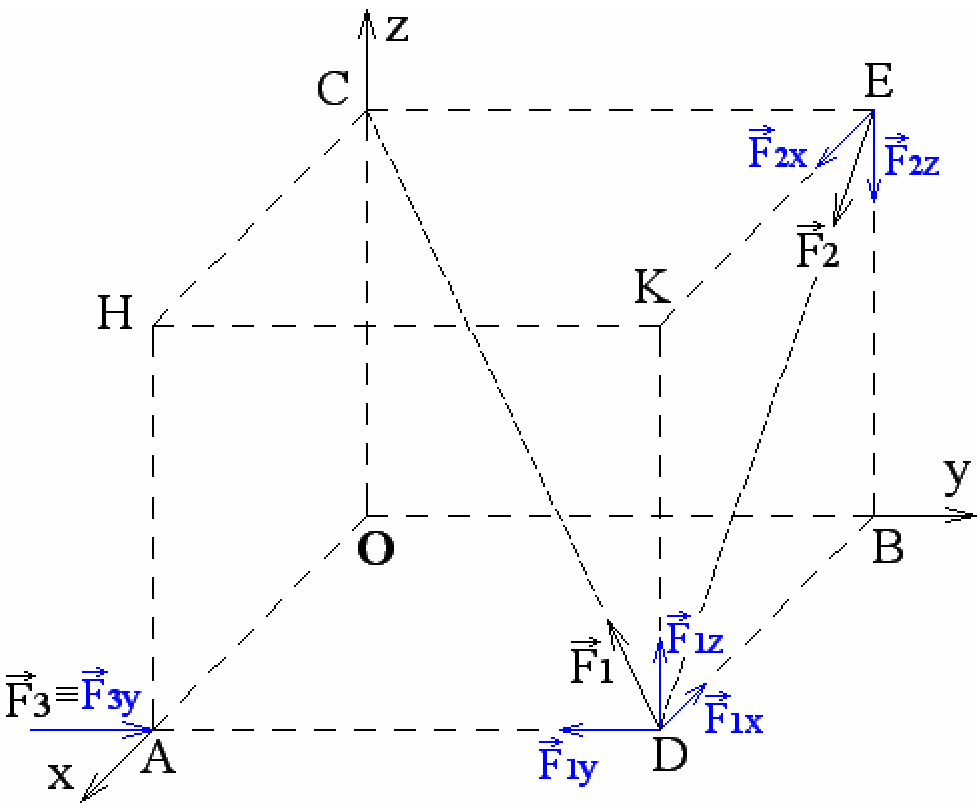


Fig. 3.1.2

- Resolution of $\vec{F}_2$

Projections of force $\vec{F}_2$ onto the axes are found using the direction cosines of vector $\overrightarrow{ED}$ collinear to the force $\vec{F}_2$, i.e.

$$\lambda_{ED} = \frac{x_D - x_E}{|ED|}; \qquad \mu_{ED} = \frac{y_D - y_E}{|ED|}; \qquad \nu_{ED} = \frac{z_D - z_E}{|ED|}.$$

The magnitude of $\overrightarrow{ED}$ is:

$$|ED| = \sqrt{(x_D - x_E)^2 + (y_D - y_E)^2 + (z_D - z_E)^2} = \sqrt{(3-0)^2 + (5-5)^2 + (0-4)^2} = 5\,\text{m},$$

and the direction cosines are:

$$\lambda_{ED} = \frac{3-0}{5} = 0{,}6; \qquad \mu_{ED} = \frac{5-5}{5} = 0; \qquad \nu_{ED} = \frac{0-4}{5} = -0{,}8.$$

- Check of the direction cosines:

$$\lambda_{ED}^2 + \mu_{ED}^2 + \nu_{ED}^2 = 1 \quad \Rightarrow \quad 0{,}6^2 + 0^2 + (-0{,}8)^2 = 1 \quad \Rightarrow \quad 1 = 1\ !$$

Projections of $\vec{F}_2$ are obtained, as follows:

$F_{2x} = \lambda_{ED} F_2 = 0{,}6.32 = 19{,}2$ kN; $F_{2y} = \mu_{ED} F_2 = 0$; $F_{2z} = \nu_{ED} F_2 = -0{,}8.32 = -25{,}6$ kN.

- Resolution of $\vec{F}_3$

Force $\vec{F}_3$ is parallel to y-axis, i.e. it has projection onto y-axis only (Fig.3.1.2):

$F_{3x} = 0$; $F_{3y} = F_3 = 28$ kN; $F_{3z} = 0$.

Finally, projections of the force resultant onto axes are carried out as:

$$R_x = -|F_{1x}| + |F_{2x}| + |F_{3x}| = -23{,}76 + 19{,}2 + 0 = -4{,}56 \text{ kN};$$

$$R_y = -|F_{1y}| + |F_{2y}| + |F_{3y}| = -39{,}6 + 0 + 28 = -11{,}6 \text{ kN};$$

$$R_z = |F_{1z}| - |F_{2z}| + |F_{3z}| = 31{,}68 - 25{,}6 + 0 = 6{,}08 \text{ kN}.$$

The magnitude of the force resultant is obtained as:

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} = \sqrt{(-4{,}56)^2 + (-11{,}6)^2 + 6{,}08^2} = 13{,}87 \text{ kN}.$$

2. Moment resultant about point O

Moment resultant $\vec{M}_O$ will be obtained by the sum of projections onto axes x, y, and z, i.e.

$$\vec{M}_O = M_{Ox}\vec{i} + M_{Oy}\vec{j} + M_{Oz}\vec{k}.$$

Here, the projections of the moments $\vec{M}_1$ and $\vec{M}_2$ onto the axes are determined first, as follows. Moment $\vec{M}_1$ is parallel to x-axis, i.e. it has projection onto x only. In contrast to $\vec{M}_1$, the moment of the couple $\vec{M}_2$ is in general position with respect to the axes x, y, z (Fig.3.1.3). The couple lies in the plane formed by points A, B, C, and vector $\vec{M}_2$ is collinear to the normal vector $\vec{N}_{ABC}$ of such plane (Fig.3.1.3). Therefore, the projections of $\vec{M}_2$ can be obtained using the direction cosines of $\vec{N}_{ABC}$.

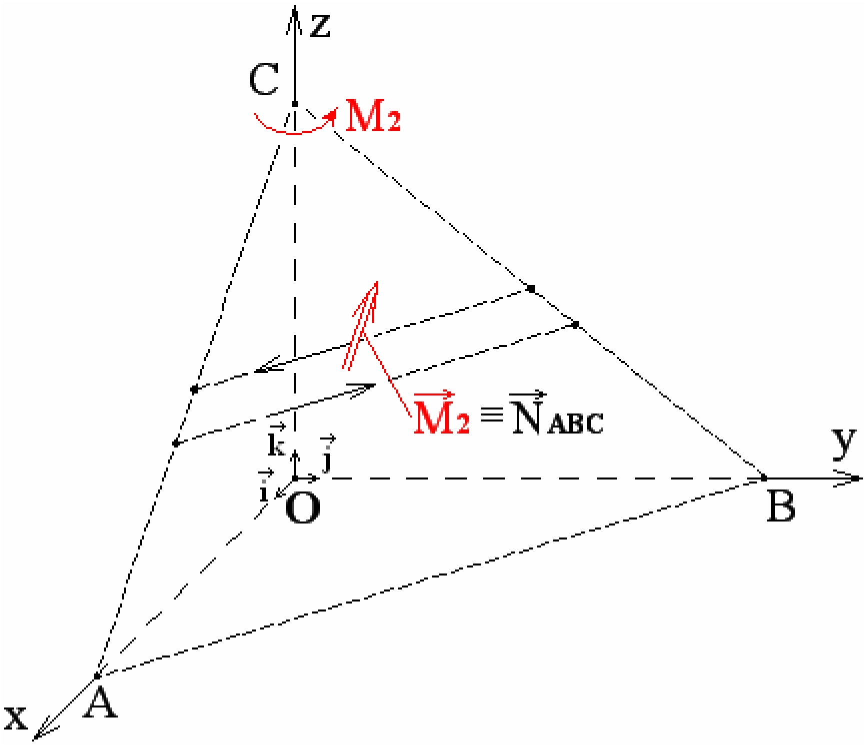


Fig. 3.1.3

The direction cosines of $\vec{N}_{ABC}$ are calculated using the cross product of vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$, two vectors lying in the plane ABC:

$$\overrightarrow{CA} \times \overrightarrow{CB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_A - x_C & y_A - y_C & z_A - z_C \\ x_B - x_C & y_B - y_C & z_B - z_C \end{vmatrix}.$$

It should be mentioned that the cross product is $\overrightarrow{CA}\times\overrightarrow{CB}$, and not $\overrightarrow{CB}\times\overrightarrow{CA}$, because during the rotation about point C, the moment $\vec{M}_2$ first crosses vector $\overrightarrow{CA}$, and then $\overrightarrow{CB}$ (Fig.3.1.3).

$$\overrightarrow{CA}\times\overrightarrow{CB}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3-0 & 0-0 & 0-4\\ 0-0 & 5-0 & 0-4\end{vmatrix}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 0 & -4\\ 0 & 5 & -4\end{vmatrix}=20\vec{i}+12\vec{j}+15\vec{k},$$

which means that

$$\left|\vec{N}_{ABC}\right|_x=20;\ \left|\vec{N}_{ABC}\right|_y=12;\ \left|\vec{N}_{ABC}\right|_z=15.$$

Further, the magnitude of $\vec{N}_{ABC}$ is calculated as:

$$\left|\vec{N}_{ABC}\right|=\sqrt{\left|\vec{N}_{ABC}\right|_x^2+\left|\vec{N}_{ABC}\right|_y^2+\left|\vec{N}_{ABC}\right|_z^2}=\sqrt{20^2+12^2+15^2}=27{,}73\,\text{m}^2.$$

Then, the direction cosines are:

$$\lambda_{ABC}=\frac{\left|\vec{N}_{ABC}\right|_x}{\left|\vec{N}_{ABC}\right|}=\frac{20}{27{,}73}=0{,}7212;\ \mu_{ABC}=\frac{\left|\vec{N}_{ABC}\right|_y}{\left|\vec{N}_{ABC}\right|}=\frac{12}{27{,}73}=0{,}4327;\ \nu_{ABC}=\frac{\left|\vec{N}_{ABC}\right|_z}{\left|\vec{N}_{ABC}\right|}=\frac{15}{27{,}73}=0{,}5409.$$

- Check of the direction cosines:

$$\lambda_{ABC}^2+\mu_{ABC}^2+\nu_{ABC}^2=1 \quad\Rightarrow\quad 0{,}7212^2+0{,}4327^2+0{,}5409^2=1 \quad\Rightarrow\quad 1=1\ !$$

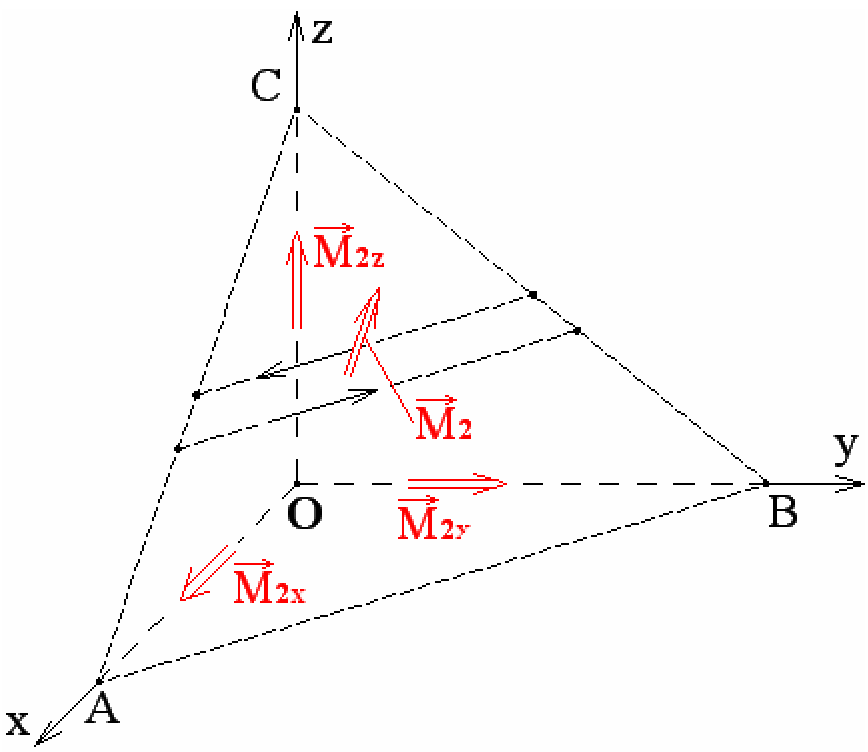


Fig. 3.1.4

Finally, the projections of $\vec{M}_2$ onto the axes x, y, z are obtained as:

$M_{2x}=\lambda_{ABC}M_2=0{,}7212.62=44{,}71\,\text{kNm};$

$M_{2y}=\mu_{ABC}M_2=0{,}4327.62=26{,}83\,\text{kNm};$

$M_{2z}=\nu_{ABC}M_2=0{,}5409.62=33{,}54\,\text{kNm},$

and their senses and directions are shown in Fig.3.1.4.

Then, the moment equations about axes x, y, z are written, as follows:

$M_{Ox}=-M_{1x}+M_{2x}+F_{1z}.\overline{AD}-F_{2z}.\overline{CE}=-25+44{,}71+31{,}68.5-25{,}6.5=50{,}11\,\text{kNm};$

$M_{Oy}=M_{2y}-F_{1z}.\overline{BD}+F_{2x}.\overline{BE}=26{,}83-31{,}68.3+19{,}2.4=8{,}59\,\text{kNm};$

$$M_{Oz} = M_{2z} + F_{1x}.\overline{AD} - F_{1y}.\overline{BD} - F_{2x}.\overline{CE} + F_{3y}.\overline{OA} = 33{,}54 + 23{,}76.5 - 39{,}6.3 - 19{,}2.5 + 28.3 = 21{,}54 \text{ kNm}$$

Using the right-hand rule, the positive senses of the projections of resultant moment are chosen to coincide with the positive senses of the axes (Fig.3.1.5).

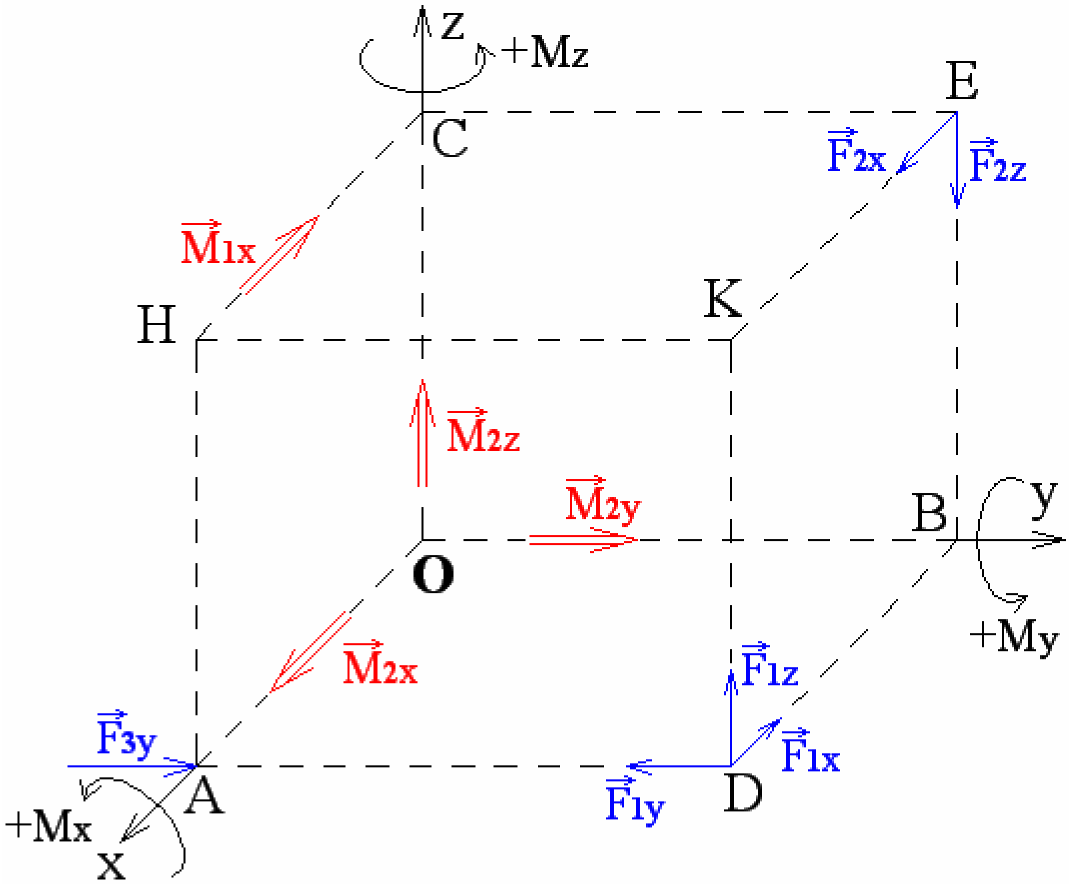


Fig. 3.1.5

Finally, the magnitude of $\vec{M}_O$ is calculated as:

$$M_O = \sqrt{M_{Ox}^2 + M_{Oy}^2 + M_{Oz}^2} = \sqrt{50{,}11^2 + 8{,}59^2 + 21{,}54^2} = 55{,}22 \text{ kNm}.$$

3. Case of reduction

The case of reduction is found using the scalar product of the force resultant and moment resultant:

$$\vec{R}\,.\,\vec{M}_O = R.M_O.\cos\beta = R_x.M_{Ox} + R_y.M_{Oy} + R_z.M_{Oz},$$

where β is the angle between directions of the force resultant and moment resultant.

Here, both $\vec{R}$ and $\vec{M}_O$ have been obtained different than zero. Therefore, if their scalar product is also different than zero, i.e. angle β is different than 90^0, then, the given set of forces can be further reduced to a wrench. However, if their dot product is equal to zero, i.e. the angle β is 90^0, then, the set of forces can be further reduced to a single resultant force.

By calculation, it is obtained:

$$\vec{R}\,.\,\vec{M}_O = (-4{,}56).50{,}11 + (-11{,}6).8{,}59 + 6{,}08.21{,}54 = -197{,}18 \text{ kN}^2\text{m}.$$

Therefore, the conclusion is that the case of reduction is wrench.

Further, to compute the angle between directions of the force resultant and moment resultant the previous expression rearranged with respect to $\cos\beta$ is used:

$$\cos\beta = \frac{R_x.M_{Ox} + R_y.M_{Oy} + R_z.M_{Oz}}{R.M_O} = \frac{-197{,}18}{13{,}87.55{,}22} = \frac{-197{,}18}{765{,}9} = -0{,}2574 \Rightarrow \beta = 104{,}92^0.$$

The force resultant, the moment resultant and the angle between them are shown in Fig.3.1.6.

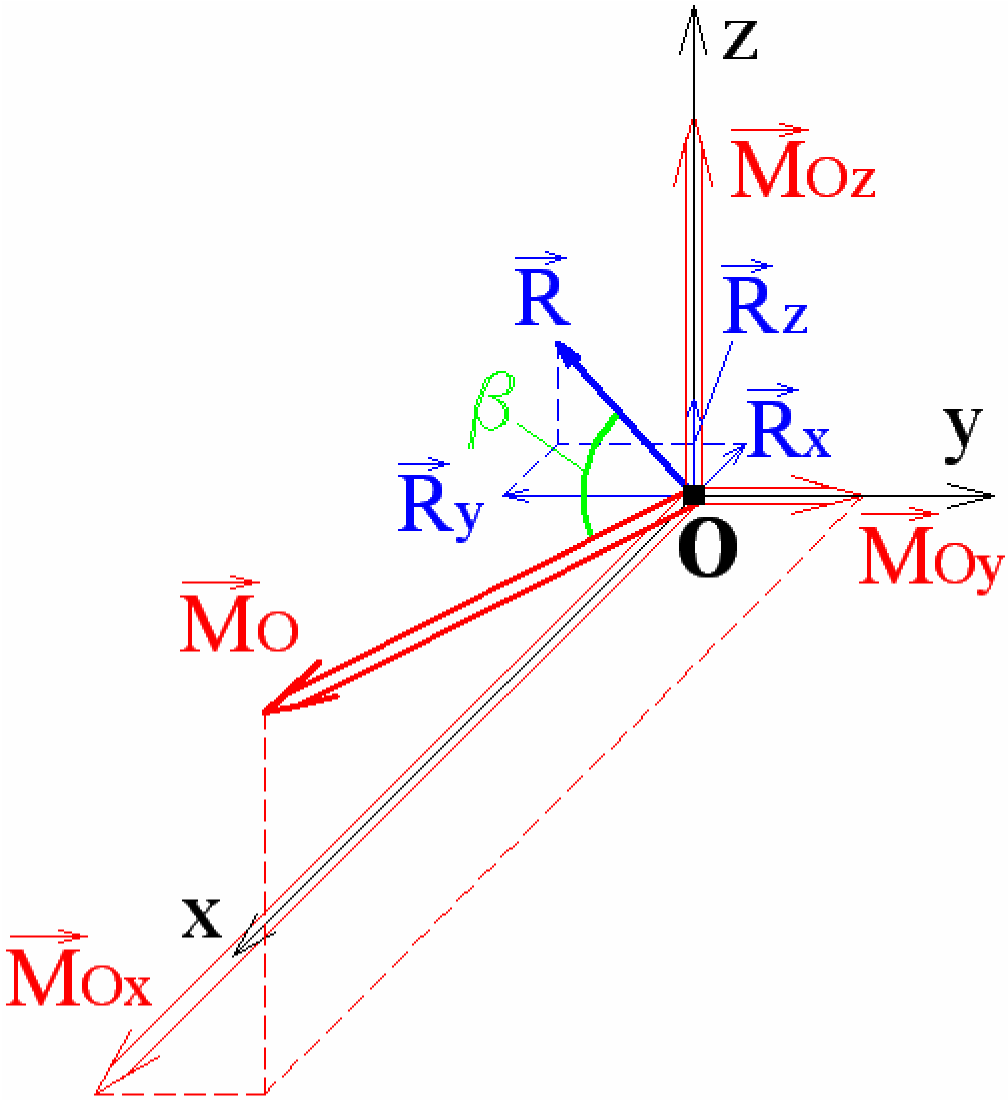


Fig. 3.1.6

4. Moment of force $\vec{F}_1$ about axis EH

The moment of a force about an axis is a scalar, equal to the projection onto the axis of the moment of the force about a point belonging to the axis. Here, point E is chosen as a point of the axis. Then:

$$M_{\vec{F}_1}^{EH} = \vec{e}_{EH} \cdot (\overrightarrow{ED} \times \vec{F}_1) = \begin{vmatrix} \lambda_{EH} & \mu_{EH} & \nu_{EH} \\ x_D - x_E & y_D - y_E & z_D - z_E \\ F_{1x} & F_{1y} & F_{1z} \end{vmatrix},$$

where $\vec{e}_{EH}$ is the unit vector of axis EH (Fig.3.1.7), while λ_{EH}, μ_{EH}, and ν_{EH} are the direction cosines of the axis EH . They are found by the following expressions:

$$\lambda_{EH} = \frac{x_H - x_E}{|EH|}; \qquad \mu_{EH} = \frac{y_H - y_E}{|EH|}; \qquad \nu_{EH} = \frac{z_H - z_E}{|EH|},$$

where $|EH| = \sqrt{(x_H - x_E)^2 + (y_H - y_E)^2 + (z_H - z_E)^2} = \sqrt{(3-0)^2 + (0-5)^2 + (4-4)^2} = 5{,}831\,\text{m}$.

$$\lambda_{EH} = \frac{3-0}{5{,}831} = 0{,}5145; \qquad \mu_{EH} = \frac{0-5}{5{,}831} = -0{,}8575; \qquad \nu_{EH} = \frac{4-4}{5{,}831} = 0.$$

- Check of the direction cosines:

$$\lambda_{EH}^2 + \mu_{EH}^2 + \nu_{EH}^2 = 1 \;\Rightarrow\; 0{,}5145^2 + (-0{,}8575)^2 + 0^2 = 1 \;\Rightarrow\; 1 = 1 \;!$$

Finally:

$$M_{F_1}^{EH} = \begin{vmatrix} 0{,}5145 & -0{,}8575 & 0 \\ 3-0 & 5-5 & 0-4 \\ -23{,}76 & -39{,}6 & 31{,}68 \end{vmatrix} = \begin{vmatrix} 0{,}5145 & -0{,}8575 & 0 \\ 3 & 0 & -4 \\ -23{,}76 & -39{,}6 & 31{,}68 \end{vmatrix} = -81{,}5\,\text{kNm}.$$

$M_{F_1}^{EH}$ has a negative sign, which means that the sense of $M_{F_1}^{EH}$ is opposite to the positive sense of axis EH (Fig.3.1.7).

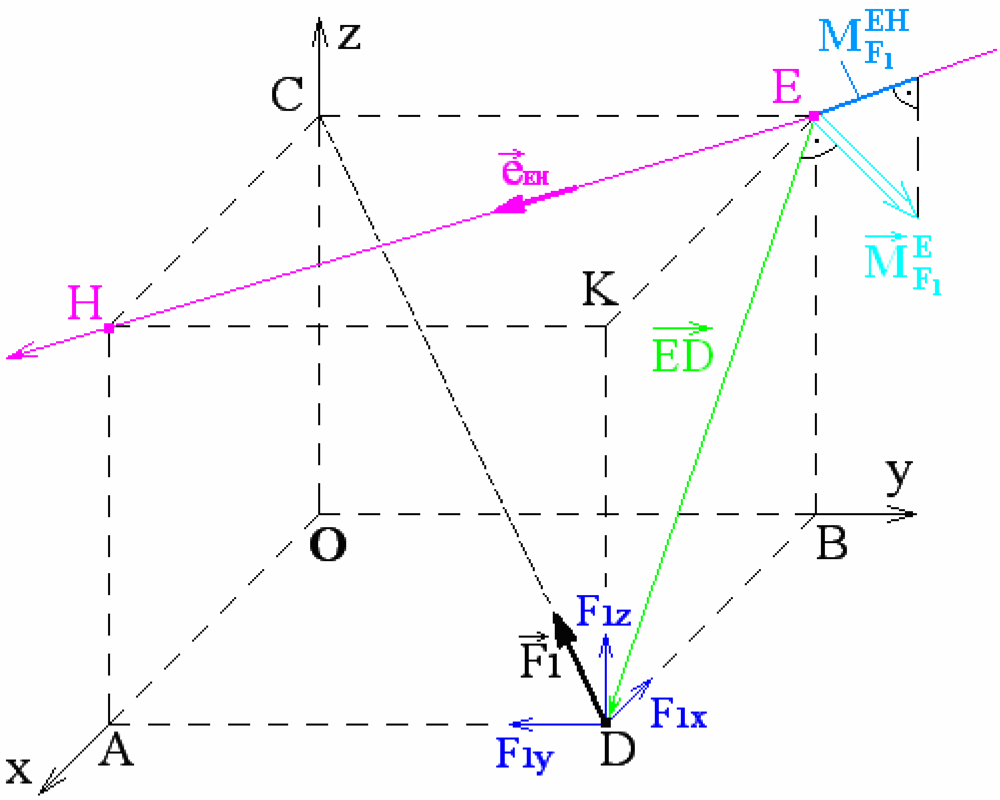


Fig. 3.1.7

COURSE WORK 4: REDUCTION OF A SET OF COPLANAR FORCES

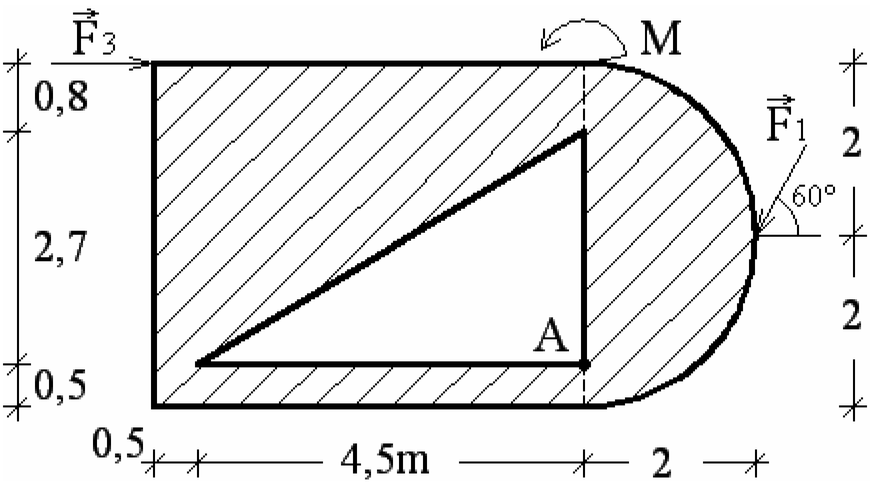
Problem 4.1

A coplanar set of forces is applied to homogeneous figure, as shown in Fig.4.1.1.

1. Locate the position of the center of gravity, point *C*;

2. Apply vertical force $\vec{F}_2$ of downward sense at point *C* and reduce the set of forces with respect to point A;

3. Determine the reduction case to which the set can be brought.



$F_1 = 30\,\text{kN}$
$F_2 = 24\,\text{kN}$
$F_3 = 36\,\text{kN}$
$M = 42\,\text{kNm}$

Fig. 4.1.1

Solution

1. Center of gravity

To obtain the coordinates of the center of gravity the following equations are used:

$$x_C = \frac{\sum_{i=1}^{n} A_i x_i}{\sum_{i=1}^{n} A_i}; \qquad y_C = \frac{\sum_{i=1}^{n} A_i y_i}{\sum_{i=1}^{n} A_i},$$

where x_i and y_i are the coordinates of the centers of gravity of the simple figures with respect to the axes, chosen for location of the center of gravity of the figure given. Here, the simple figures are three – rectangle, triangle, semicircle, and, first, the absolute location of their centers of gravity and their areas have to be determined.

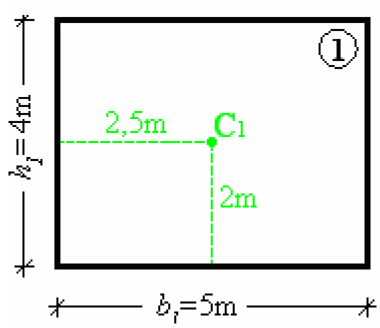


Fig. 4.1.2a

Figure (1) is rectangle. The coordinates of the center of gravity are:

$C_1(\frac{h_1}{2} = 2\,\text{m}; \frac{b_1}{2} = 2{,}5\,\text{m})$ – (Fig.4.1.2a).

The area of the rectangle is:

$A_1 = 5.4 = 20\,\text{m}^2.$

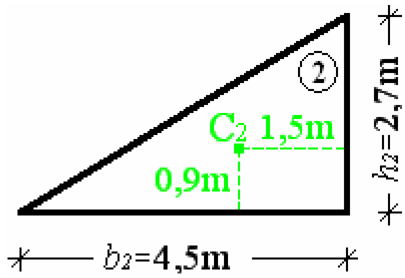


Fig. 4.1.2b

Figure (2) is right-angled triangle and the coordinates of the center of gravity with respect to the right angle are:

$$C_2(\frac{h_2}{3}=0{,}9\,\mathrm{m};\frac{b_2}{3}=1{,}5\,\mathrm{m})-(\mathrm{Fig.4.1.2b}).$$

The area of the triangle is:

$$A_2=\frac{1}{2}.2{,}7.4{,}5=6{,}075\,\mathrm{m}^2.$$

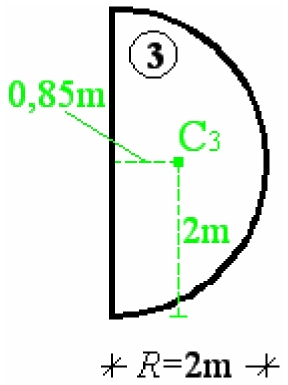


Fig. 4.1.2c

Figure (3) is semicircle. The coordinates of the center of gravity are:

$$C_3(R=2\,\mathrm{m};\frac{4R}{3\pi}=0{,}85\,\mathrm{m})-(\mathrm{Fig.4.1.2c}).$$

The area of the semicircle is:

$$A_3=\frac{1}{2}.\pi.R^2=\frac{1}{2}.3{,}14.2^2=6{,}28\,\mathrm{m}^2.$$

After that, the coordinates of the centers of gravity of three figures about axes x and y have to be found. The axes x and y are chosen so that the entire figure lies in the first quadrant (Fig.4.1.3). Thus, all of the coordinates of the centers of gravity of the simple figures are positive.

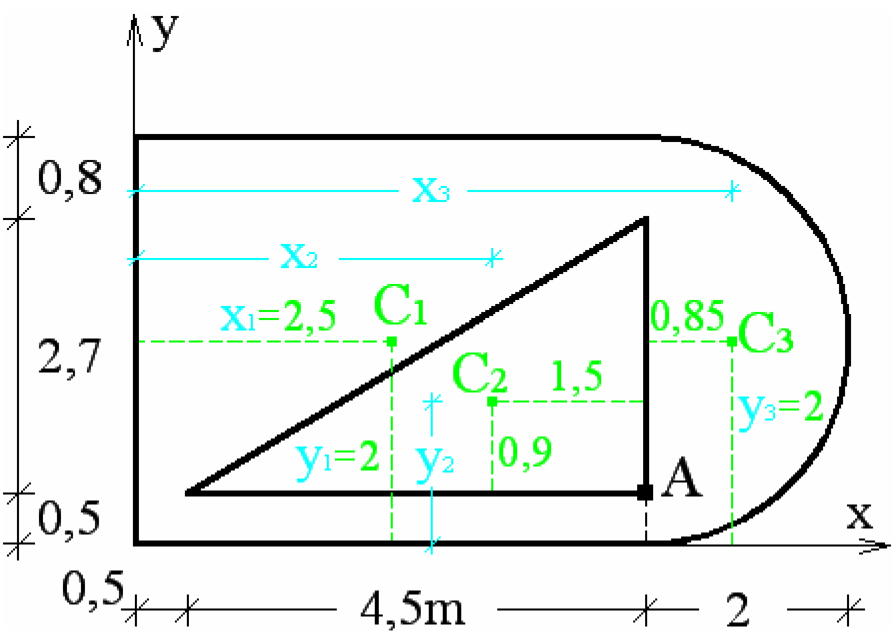


Fig. 4.1.3

Using the coordinates determined earlier and following Fig.4.1.3, it is obtained:

$x_1=2{,}5\,\mathrm{m};$

$y_1=2\,\mathrm{m};$

$x_2=0{,}5+4{,}5-1{,}5=3{,}5\,\mathrm{m};$

$y_2=0{,}9+0{,}5=1{,}4\,\mathrm{m};$

$x_3=0{,}5+4{,}5+0{,}85=5{,}85\,\mathrm{m};$

$y_3=2\,\mathrm{m},$

i.e. the locations of the centers of gravity about x and y are:

$C_1(2{,}5\,\mathrm{m};2\,\mathrm{m});$

$C_2(3{,}5\,\mathrm{m};1{,}4\,\mathrm{m});$

$C_3(5{,}85\,\mathrm{m};2\,\mathrm{m}).$

Finally, the coordinates of the center of gravity of the given figure about axes x and y are:

$$x_C=\frac{\sum_{i=1}^{3}A_ix_i}{\sum_{i=1}^{3}A_i}=\frac{A_1x_1-A_2x_2+A_3x_3}{A_1-A_2+A_3}=\frac{20.2{,}5-6{,}075.3{,}5+6{,}28.5{,}85}{20-6{,}075+6{,}28}=\frac{65{,}475}{20{,}205}=3{,}24\,\mathrm{m};$$

$$y_C = \frac{\sum_{i=1}^{3} A_i y_i}{\sum_{i=1}^{3} A_i} = \frac{A_1 y_1 - A_2 y_2 + A_3 y_3}{A_1 - A_2 + A_3} = \frac{20.2 - 6{,}075.1{,}4 + 6{,}28.2}{20 - 6{,}075 + 6{,}28} = \frac{44{,}055}{20{,}205} = 2{,}18\,\text{m}.$$

and its position is given in Fig. 4.1.4.

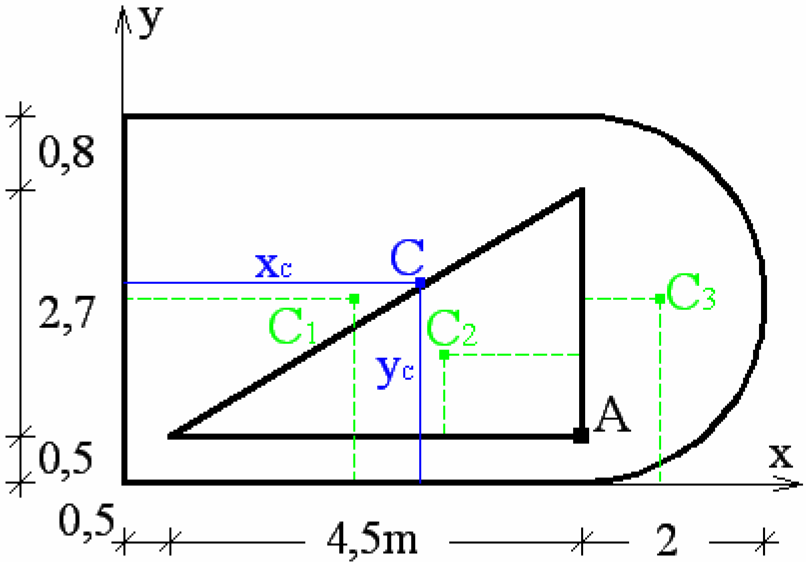


Fig. 4.1.4

2. Reduction of the set of forces about point A

After the center of gravity has already been located, the force $\vec{F}_2$ is applied (Fig.4.1.5) and the reduction of the set with respect to the point A begins.

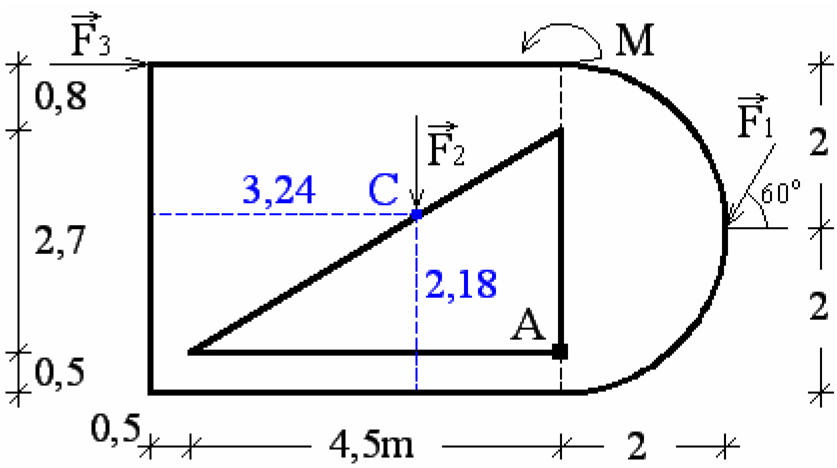


Fig. 4.1.5

2.1 Force resultant

First, the axes X and Y of origin point A are introduced (Fig.4.1.6). Then, the force resultant is:

$$\vec{R} = \vec{R}_X + \vec{R}_Y,$$

$$\vec{R}_u = \vec{F}_{1X} + \vec{F}_{2X} + \vec{F}_{3X},$$

$$\vec{R}_v = \vec{F}_{1Y} + \vec{F}_{2Y} + \vec{F}_{3Y}.$$

Here, $\vec{F}_{1X}$, ……., $\vec{F}_{3Y}$ are projections of the applied forces onto axes X and Y (Fig.4.1.6):

$F_{1X} = F_1 \cos 60^0 = 30.0{,}5 = 15\,\text{kN}$; $F_{1Y} = F_1 \sin 60^0 = 30.0{,}866 = 26\,\text{kN}$;

$F_{2X} = 0$; $F_{2Y} = F_2 = 24\,\text{kN}$; $F_{3X} = F_3 = 36\,\text{kN}$; $F_{3Y} = 0$.

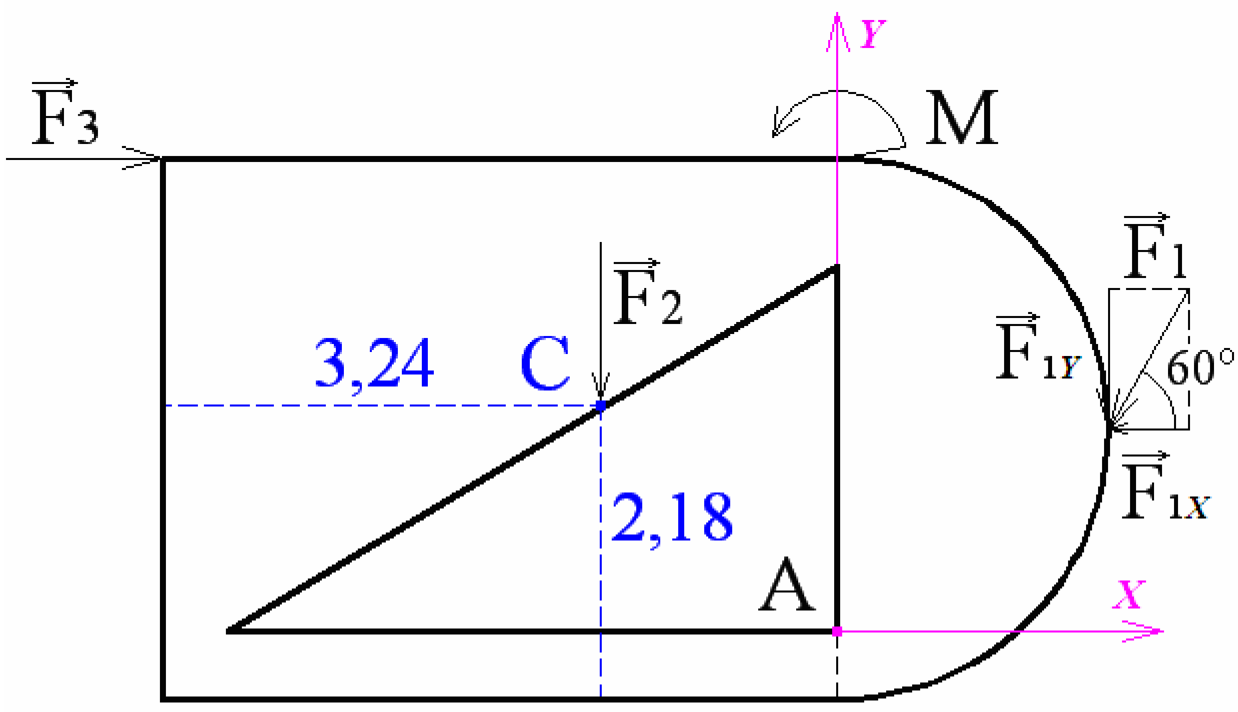


Fig. 4.1.6

Further:

$$R_X = -F_{1X} + F_3 = -15 + 36 = 21\,\text{kN};$$

$$R_Y = -F_{1Y} - F_2 = -26 - 24 = -50\,\text{kN};$$

$$R = \sqrt{R_X^2 + R_Y^2} = \sqrt{21^2 + (-50)^2} = 54{,}23\,\text{kN}.$$

Finally, the angle between X-axis and resultant force is calculated. The result is:

$$tg\alpha_R = \frac{R_Y}{R_X} = \frac{-50}{21} = -2{,}381 \;\Rightarrow\qquad \alpha_R = -67{,}22^0 \text{ (Fig.4.1.7).}$$

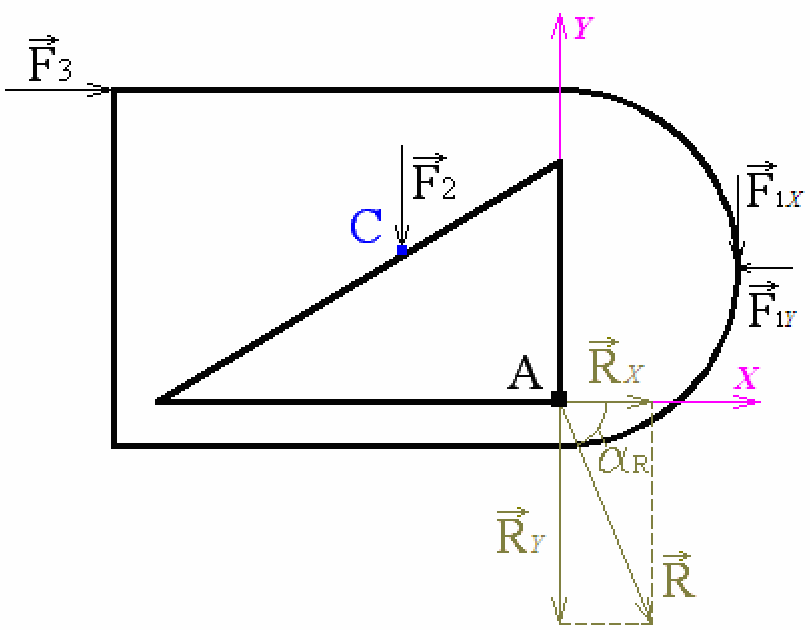


Fig. 4.1.7

2.2 Moment resultant

The moment resultant about point A is obtained by the following expression:

$$M_A = M + F_{1X}.(2-0{,}5) - F_{1Y}.2 + F_2.(5-3{,}24) - F_3.3{,}5 = 42 + 15.1{,}5 - 26.2 + 24.1{,}76 - 36.3{,}5 = -71{,}26\,\text{kNm},$$

where the positive sense of the moment is counterclockwise (Fig.4.1.8). The result is obtained negative meaning that the moment resultant points toward the page (Fig.4.1.9).

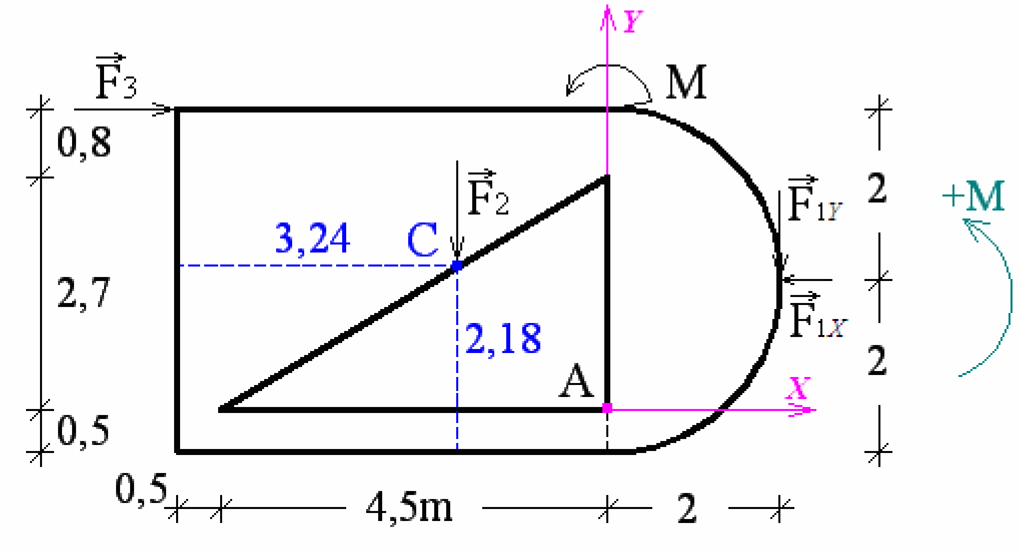


Fig. 4.1.8

2.3 Case of reduction

The set of forces is coplanar, which means that it can be further reduced to the single resultant force (Fig. 4.1.9). The equation of the direction of single resultant force is:

$$XR_Y - YR_X = M_A;$$

$$X.(-50) - Y.21 = -71{,}26;$$

$$n_X = \frac{M_A}{R_Y} = \frac{-71{,}26}{-50} = 1{,}425\,\text{m}; \qquad n_Y = \frac{M_A}{-R_X} = \frac{-71{,}26}{-21} = 3{,}393\,\text{m};$$

$$h_R = \left|\frac{M_A}{R}\right| = \left|\frac{-71{,}26}{54{,}23}\right| = 1{,}314\,\text{m}.$$

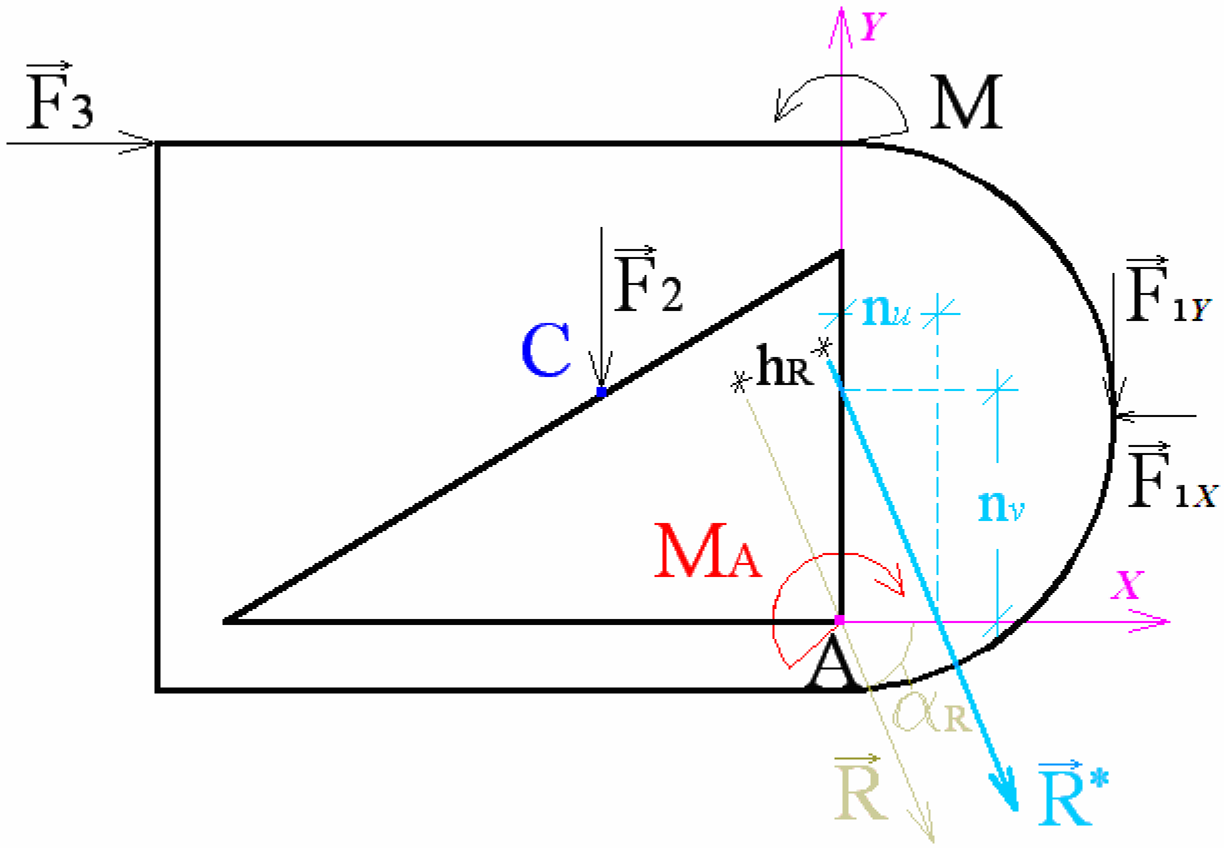


Fig. 4.1.9

COURSE WORK 5: EQUILIBRIUM OF A BODY ACTED UPON BY A SPATIAL SET OF FORCES

Problem 5.1

Determine the reactions of the rectangular prism supported and loaded as shown in Fig.5.1.1. Check the result obtained using the relevant equilibrium equations. The weight per unit volume is $\gamma = 3\,\text{kN/m}^3$.

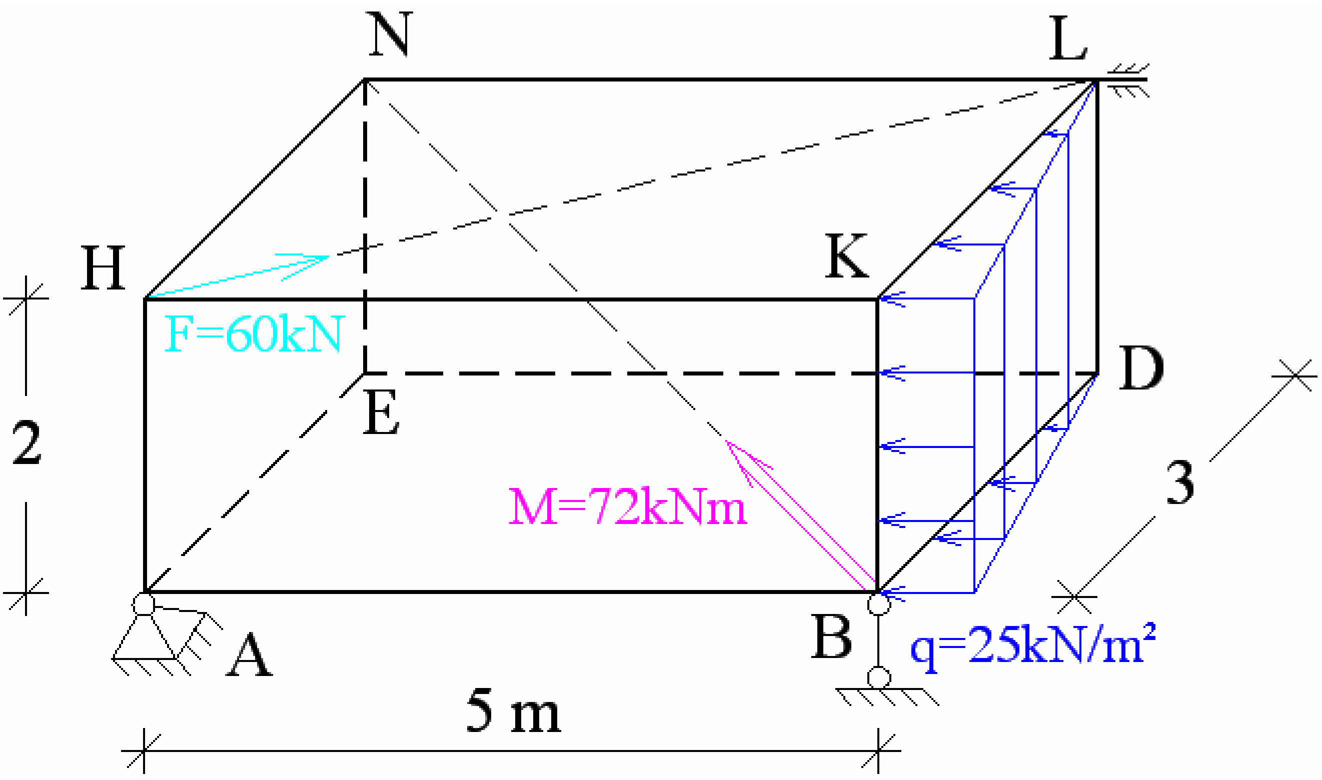


Fig. 5.1.1

Solution:

1. Determination of the weight of the body, the resultant force of the distributed load, the projections of the moment M and the force F onto the axes of the appropriate coordinate system

First, the Cartesian coordinate system xyz of origin point A is introduced. Point A has been chosen for origin, because it is a spherical support containing three unknown reactions. Thus, if the moment equations about axes x, y, z are written, then, the three reactions at point A are eliminated.

- The weight of the rectangular prism is:

$$G = V.\gamma,$$

where $V = AB.BD.DL = 5.3.2 = 30\,\text{m}^3$ is the volume. Then:

$$G = 30.3 = 90\,\text{kN}.$$

G is directed down vertical force applied at the body's center of gravity (Fig.5.1.2).

- Distributed load q is applied on the side $BDLK$ of the rectangular prism. Then, the resultant force is:

$$R_q = \frac{1}{2}BD.BK.q = \frac{1}{2}3.2.25 = 75\,\text{kN}.$$

The line of action and sense of R_q are determined by the line of action and sense of the distributed load. The point of application of R_q is the projection of the distributed load's center of gravity on the prism's side $BDLK$ (Fig.5.1.2).

- Moment M is in general position with respect to the axes x, y, z. Then, in order to find its projections onto these axes, the direction cosines of vector $\overrightarrow{BN}$, collinear to $\vec{M}$, are used (Fig.5.1.2). The coordinates of the points formed $\overrightarrow{BN}$ are $N(-3;0;2)$ and $B(0;5;0)$. Then, the magnitude of $\overrightarrow{BN}$ is:

$$\left|\overrightarrow{BN}\right| = \sqrt{(x_N - x_B)^2 + (y_N - y_B)^2 + (z_N - z_B)^2} = \sqrt{(-3-0)^2 + (0-5)^2 + (2-0)^2} = 6{,}164 \text{ m}.$$

Further,

$$\lambda_{BN} = \frac{x_N - x_B}{\left|\overrightarrow{BN}\right|} = \frac{-3}{6{,}164} = -0{,}4867\,; \quad \mu_{BN} = \frac{y_N - y_B}{\left|\overrightarrow{BN}\right|} = \frac{-5}{6{,}164} = -0{,}8111\,; \quad \nu_{BN} = \frac{z_N - z_B}{\left|\overrightarrow{BN}\right|} = \frac{2}{6{,}164} = 0{,}3244\,.$$

- Check of the direction cosines:

$\lambda_{BN}^2 + \mu_{BN}^2 + \nu_{BN}^2 = 1;\ (-0{,}4867)^2 + (-0{,}8111)^2 + 0{,}3244^2 = 1;\ 0{,}9999 \approx 1\,!$

Finally, the projections of the moment are found as:

$M_x = \lambda_{BN}.M = -0{,}4867.72 = -35{,}04$ kNm; $M_y = \mu_{BN}.M = -0{,}8111.72 = -58{,}4$ kNm;

$M_z = \nu_{BN}.M = 0{,}3244.72 = 23{,}36$ kNm.

- Force F lies on the side $KLNH$ of the body. Then, it has the projections onto axes x and y:

  $F_x = F.\cos\beta$; $\qquad F_y = F.\sin\beta$ (Fig.5.1.2) .

Considering right-angled triangle LNH , angle β is obtained as:

$$tg\beta = \frac{LN}{NH} = \frac{5}{3} \Rightarrow \beta = 59{,}04^0\,.$$

Then,

$F_x = 60.\cos 59{,}04^0 = 30{,}9$ kN; $\qquad F_y = 60.\sin 59{,}04^0 = 51{,}45$ kN (Fig.5.1.2).

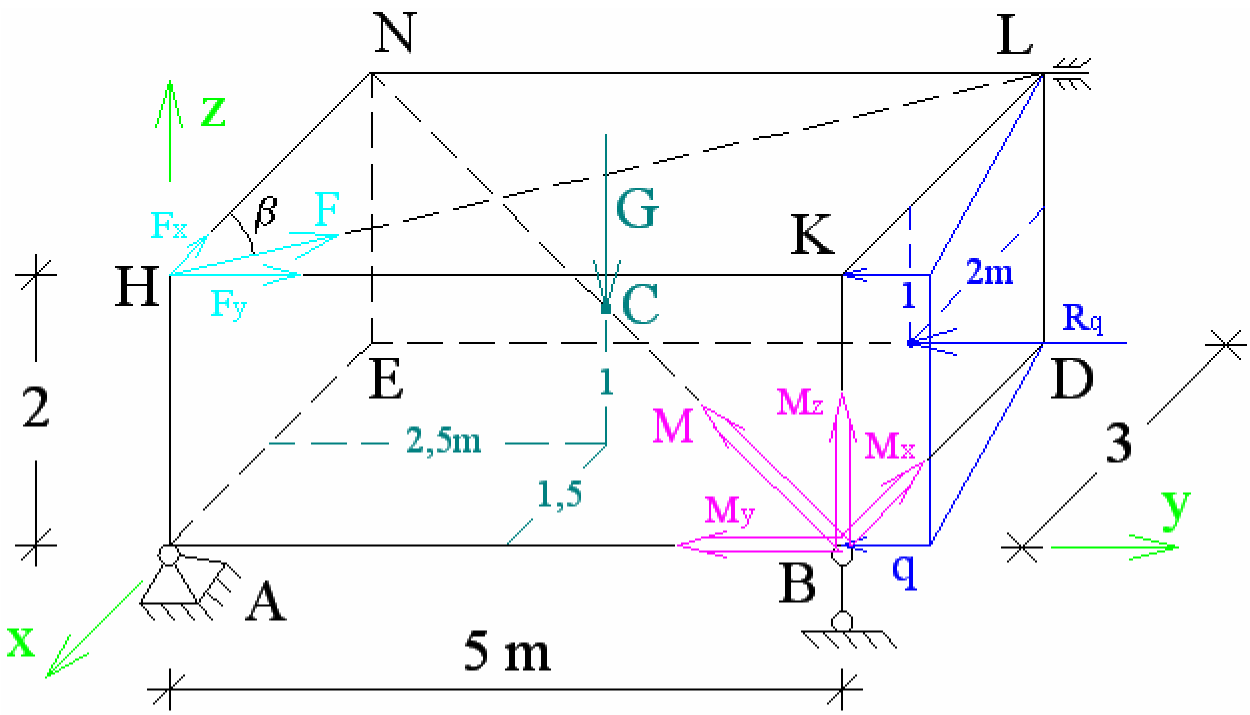


Fig. 5.1.2

2. Determination of support reactions

First, free body diagram is drawn, i.e. all supports are detached from the body and the support reactions are introduced instead. These are:

- Point A is a spherical support constraining the displacements parallel to the axes of the coordinate system. Then, three support reactions are available, namely A_x, A_y, A_z where their senses are supposed (Fig.5.1.3);
- There is a link at point B which means that only one support reaction of line of action coinciding with the direction of the link has to be introduced (Fig.5.1.3);
- Point L is a cylindrical support and contains two support reactions, L_x and L_z, of lines of action parallel to the axes x and z (Fig.5.1.3).

Besides, all external loads are applied.

The spatial set of forces is in equilibrium when the main vector and the main moment are both equal to zero, i.e. $\vec{R}=0$, $\vec{M}_O=0$. These two vector equations can be transformed into nine scalar equations, three force and six moment equations. Six of them are used for the support reactions determination, while the other three are applied for the check of the reactions obtained.

Further, in order to determine the support reactions, the appropriate choice of equations has to be made. In the best case, the independent equations in which only one unknown reaction takes part has to be written. Here, in order to find the independent equations, the free-body diagram is analyzed carefully (Fig.5.1.3).

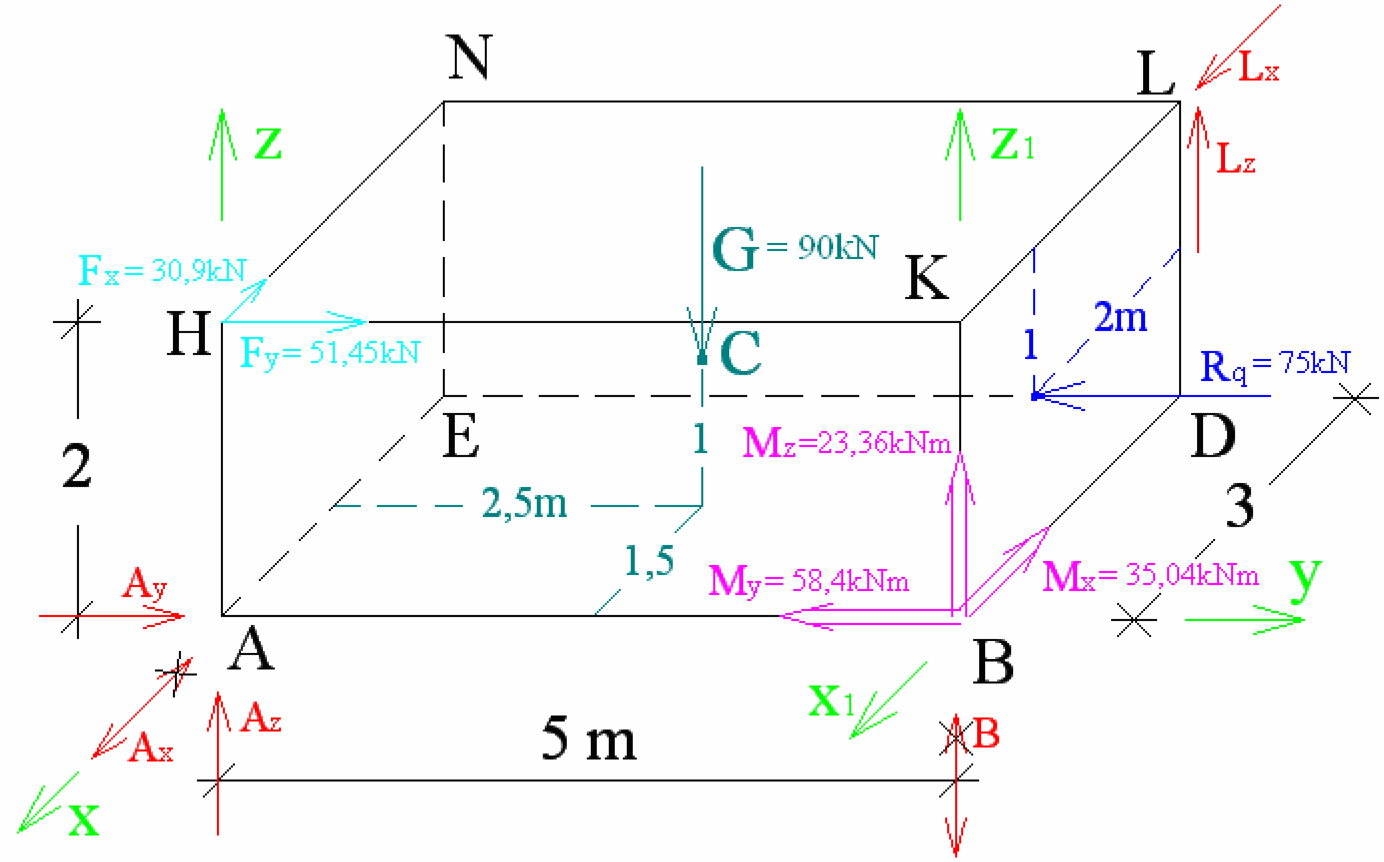


Fig. 5.1.3

It is obvious that only reaction A_y is parallel to y-axis. Then, the first independent equation is:

1) $\sum F_{iy}=0$; $A_y+F_y-R_q=0$; $A_y+51{,}45-75=0$; $A_y=23{,}55$ kN.

A_y is positive which means that the sense supposed is correct (Fig.5.1.3).

The second independent equation is:

2) $\sum M_z=0$; $M_z+R_q.1-L_x.5=0$; $23{,}36+75.1-L_x.5=0$; $L_x=\dfrac{75.1+23{,}36}{5}=19{,}67$ kN.

The third equation containing only one unknown is:

3) $\sum M_{z_1}=0$; $-A_x.5-F_x.5+M_z+R_q.1=0$; $-A_x.5-30{,}9.5+23{,}36+75.1=0$;

$$A_x=\frac{75.1+23{,}36-30{,}9.5}{5}=-11{,}23 \text{ kN}.$$

The sign "-" for A_x means that the sense supposed is not correct and has to be changed (Fig.1.3).

The solution continues with the fourth independent equation:

4) $\sum M_{x_1}=0$; $-A_z.5-F_y.2+G.2{,}5-M_x+R_q.1=0$; $-A_z.5-51{,}45.2+90.2{,}5-35{,}04+75.1=0$;

$$A_z=\frac{90.2{,}5+75.1-35{,}04-51{,}45.2}{5}=32{,}41 \text{ kN}.$$

Further, analyzing Fig.5.1.3 the conclusion that there is no other independent equation is made. Then, the determination of the last support reactions continues with the next equations:

5) $\sum M_y = 0;\ -G.1{,}5 - M_y - F_x.2 + L_z.3 + L_x.2 = 0;\ -90.1{,}5 - 58{,}4 - 30{,}9.2 + L_z.3 + 19{,}67.2 = 0;$

$$L_z = \frac{90.1{,}5 + 58{,}4 + 30{,}9.2 - 19{,}67.2}{3} = 71{,}95\,\text{kN};$$

6) $\sum M_x = 0;\ B.5 - F_y.2 - G.2{,}5 + R_q.1 - M_x + L_z.5 = 0;$

$$B.5 - 51{,}45.2 - 90.2{,}5 + 75.1 - 35{,}04 + 71{,}95.5 = 0;$$

$$B = \frac{51{,}45.2 + 90.2{,}5 - 75.1 + 35{,}04 - 71{,}95.5}{5} = -14{,}36\,\text{kN}.$$

3. Check for the reactions obtained

One force and five moment equations have been used for determination of the support reactions. Then, two force and one moment equation have to be written for the check of the result obtained. It should be noted that any equation chosen for check has to contain as many as possible support reactions.

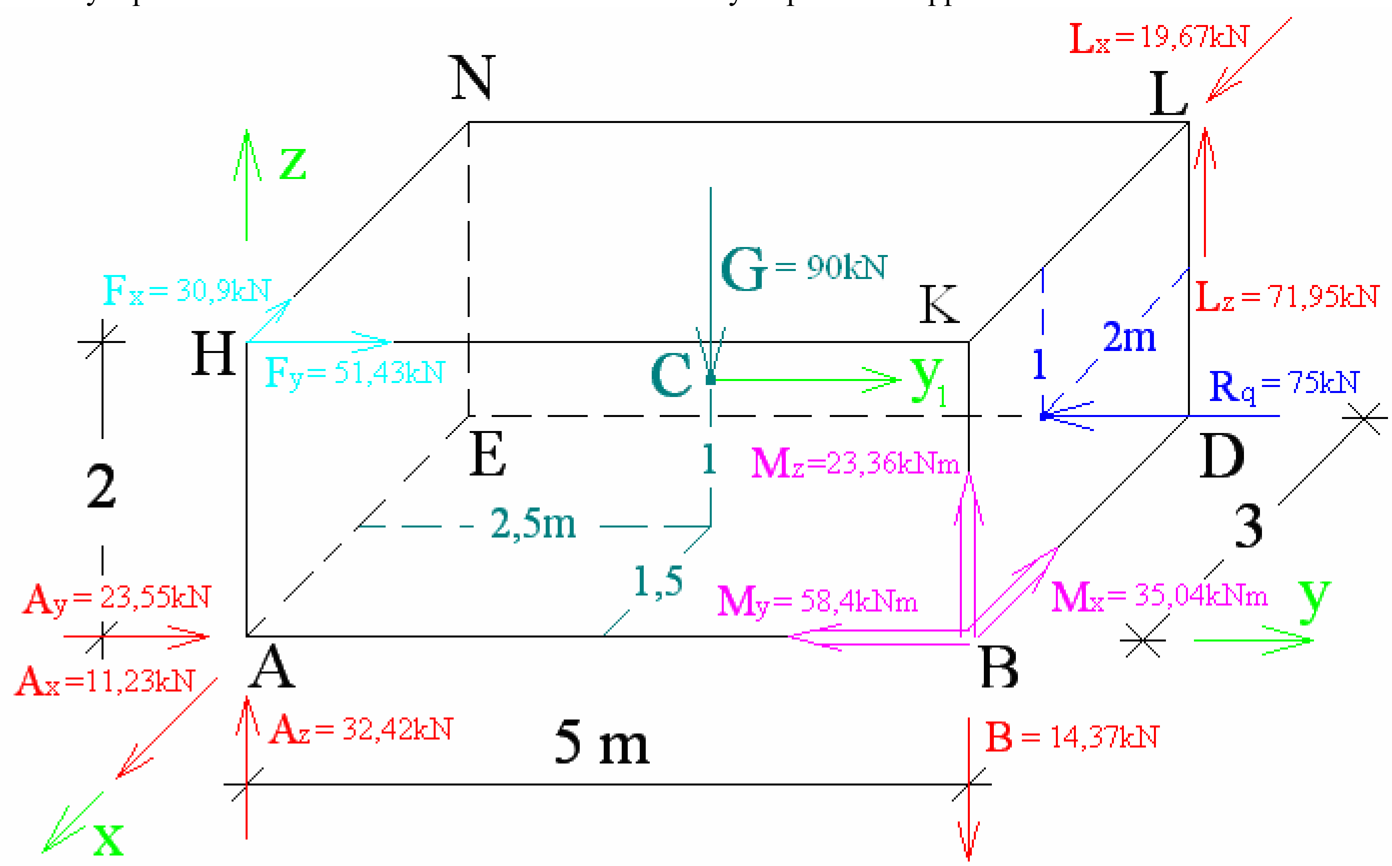


Fig. 5.1.4

The force equations for the check are (Fig.5.1.4):

7) $\sum F_{ix} = 0;\ A_x - F_x + D_x = 0;\ 11{,}23 - 30{,}9 + 19{,}67 = 0;\ 30{,}9 - 30{,}9 = 0\,!$

8) $\sum F_{iz} = 0;\ A_z - G - B + D_z = 0;\ 32{,}41 - 90 - 14{,}36 + 71{,}95 = 0;\ 104{,}36 - 104{,}36 = 0\,!$

The moment equation is (Fig.5.1.4):

9) $\sum M_{y_1} = 0;\quad -A_x.1 - A_z.1{,}5 - F_x.1 - M_y + L_x.1 + L_z.1{,}5 + B.1{,}5 = 0;$

$$-11{,}23.1 - 32{,}41.1{,}5 - 30{,}9.1 - 58{,}4 + 19{,}67.1 + 71{,}95.1{,}5 + 14{,}36.1{,}5 = 0;$$

$$149{,}135 - 149{,}145 = -0{,}01 \approx 0\,!$$

All equations are obtained equal to zero meaning that the support reactions are determined correctly!

Problem 5.2

Determine the support reactions of the rectangular plate supported and loaded as shown in Fig.5.2.1. Check the result obtained using the relevant equilibrium equations. Assume that the thickness of the plate is approximately zero. Neglect the weight of the plate.

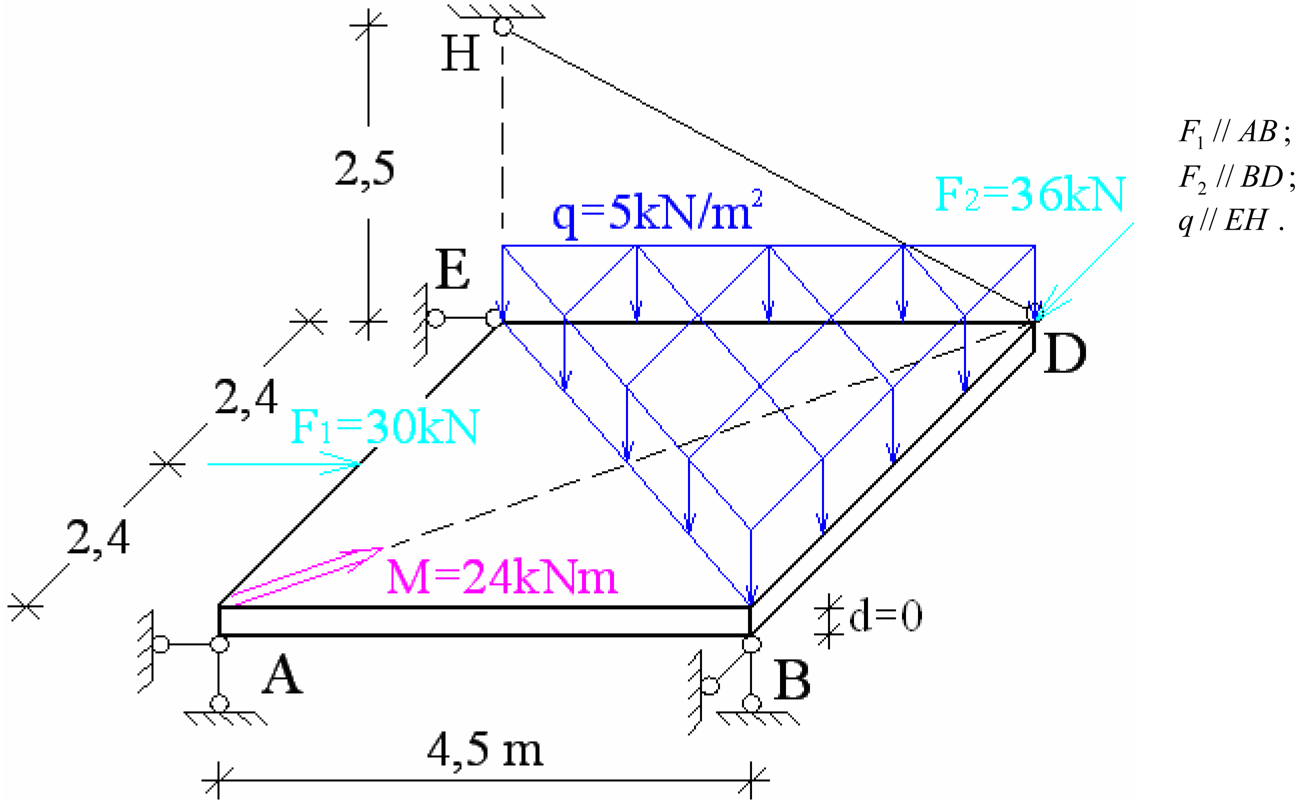


Fig. 5.2.1

Solution:

1. Determination of the resultant force of the distributed load, the projections of the moment M and the support reaction D onto the axes of the appropriate coordinate system

Solution begins with the choice of the coordinate system. Here, the coordinate system is chosen so that the axes pass through as many as possible supports (Fig.5.2.2).

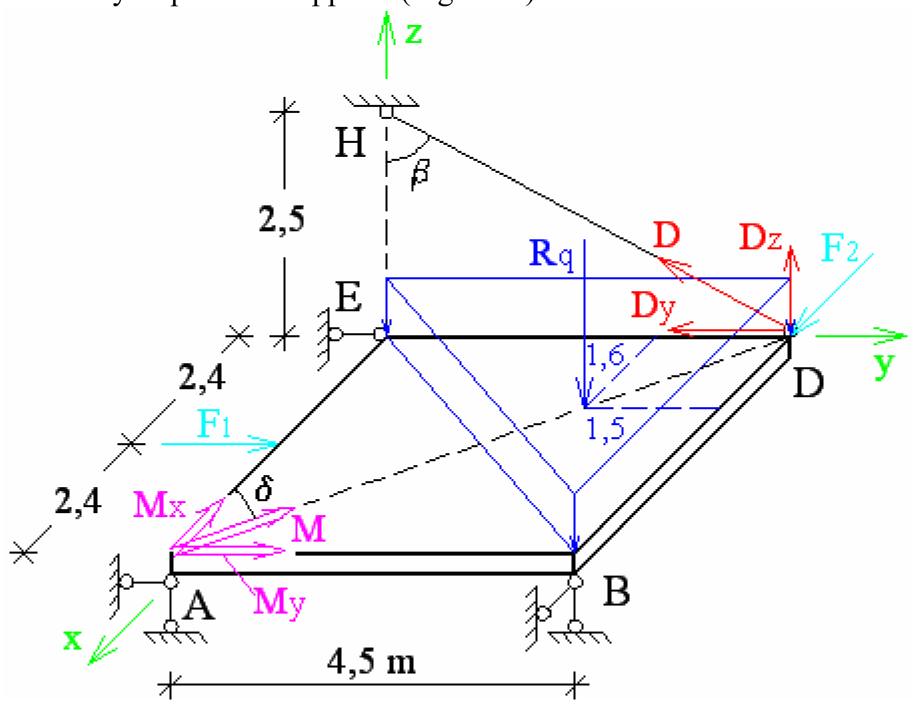


Fig. 5.2.2

- The distributed load q acts on the half plate. To determine the resultant force R_q the formula for the volume of the prism is applied, i.e.

$$R_q = \frac{1}{2} AB.BD.q = \frac{1}{2} 4{,}5.4{,}8.5 = 54 \text{ kN}.$$

R_q is a force parallel to *z*-axis of sense coinciding with the sense of q. The point of application of R_q is the projection onto the plate of the distributed load's center of gravity (Fig.5.2.2).

- Moment M has projections onto axes x and y, and they are obtained using angle δ as:

$$M_x = M.\cos\delta\,; \quad M_y = M.\sin\delta\,,$$

where δ is :

$$tg\delta = \frac{DE}{AE} = \frac{4{,}5}{4{,}8} = 0{,}9375 \;\Rightarrow\; \delta = 43{,}15^0 \text{ (Fig.5.2.2)}.$$

Then,

$$M_x = 24.\cos 43{,}15^0 = 17{,}5 \text{ kNm}; \qquad M_y = 24.\sin 43{,}15^0 = 16{,}4 \text{ kNm (Fig.5.2.2)}.$$

- The link at point D is inclined with respect to the axes *y* and *z* and the reaction in the link is in general position with respect to the same axes (Fig.5.2.2). Therefore, it is more convenient to work with the projections of the force D onto the axes *y* and *z*. These projections are determined as follows:

$$D_y = D.\sin\beta\,; \qquad D_z = D.\cos\beta\,,$$

where $tg\beta = \frac{ED}{EH} = \frac{4{,}5}{2{,}5} = 1{,}8 \;\Rightarrow\; \beta = 60{,}95^0$.

Finally:

$$D_y = D.\sin 60{,}95 = 0{,}8741D\,; \; D_z = D.\cos 60{,}95 = 0{,}4856D \text{ (Fig.5.2.2)}.$$

2. Determination of the support reactions

The plate is detached from the constraints and the support reactions are introduced instead:

- There are two links at point A and reactions A_y and A_z of lines of action coinciding with directions of the links are introduced (Fig.5.2.3);
- There are two links at point B and two reactions are introduced, as well (Fig.2.3);
- The link at point D has already been considered and the projections of the reaction, namely D_y and D_z have been introduced (Fig.5.2.3);
- There is a link at point E and one reaction is introduced (Fig.5.2.3).

It should be noted that the senses of all reactions are supposed.

Further, the solution continues with analysis of Fig.5.2.3. Similar to the previous example, the independent equations are four. These are:

1) $\sum F_{ix} = 0\,;\; -B_x + F_2 = 0\,;\; B_x + 36 = 0\,;\; B_x = 36 \text{ kN}.$

2) $\sum M_{z_1} = 0\,;\; A_y.4{,}8 + F_1.2{,}4 = 0\,;\; A_y.4{,}8 + 30.2{,}4 = 0\,;\; A_y = \frac{-30.2{,}4}{4{,}8} = -15 \text{ kN}.$

3) $\sum M_{x_1} = 0\,;\; -A_z.4{,}5 - M_x + R_q.1{,}5 = 0\,;\; -A_z.4{,}5 - 17{,}5 + 54.1{,}5 = 0\,;$

$$A_z = \frac{-17{,}5 + 54.1{,}5}{4{,}5} = 14{,}11 \text{ kN}.$$

4) $\sum M_{y_1} = 0\,; M_y - R_q.3{,}2 + D_z.4{,}8 = 0\,;\; 16{,}4 - 54.3{,}2 + D_z.4{,}8 = 0\,;$

$$D_z = \frac{-16{,}4 + 54.3{,}2}{4{,}5} = 32{,}58 \text{ kN} \Rightarrow D = \frac{32{,}58}{0{,}4856} = 67{,}09 \text{ kN} \Rightarrow D_y = 0{,}8741.67{,}09 = 58{,}64 \text{ kN}.$$

The last two equations are:

5) $\sum M_x = 0\,;\; -M_x + B_z.4{,}5 - R_q.3 + D_z.4{,}5 = 0\,;\; -17{,}5 + B_z.4{,}5 - 54.3 + 32{,}58.4{,}5 = 0\,;$

$$B_z = \frac{17{,}5 + 54.3 - 32{,}58.4{,}5}{4{,}5} = 7{,}31 \text{ kN};$$

6) $\sum M_{z_2} = 0\,;\; -E.4{,}8 - F_1.2{,}4 + D_y.4{,}8 = 0\,;\; -E.4{,}8 - 30.2{,}4 + 58{,}64.4{,}8 = 0\,;$

kN.

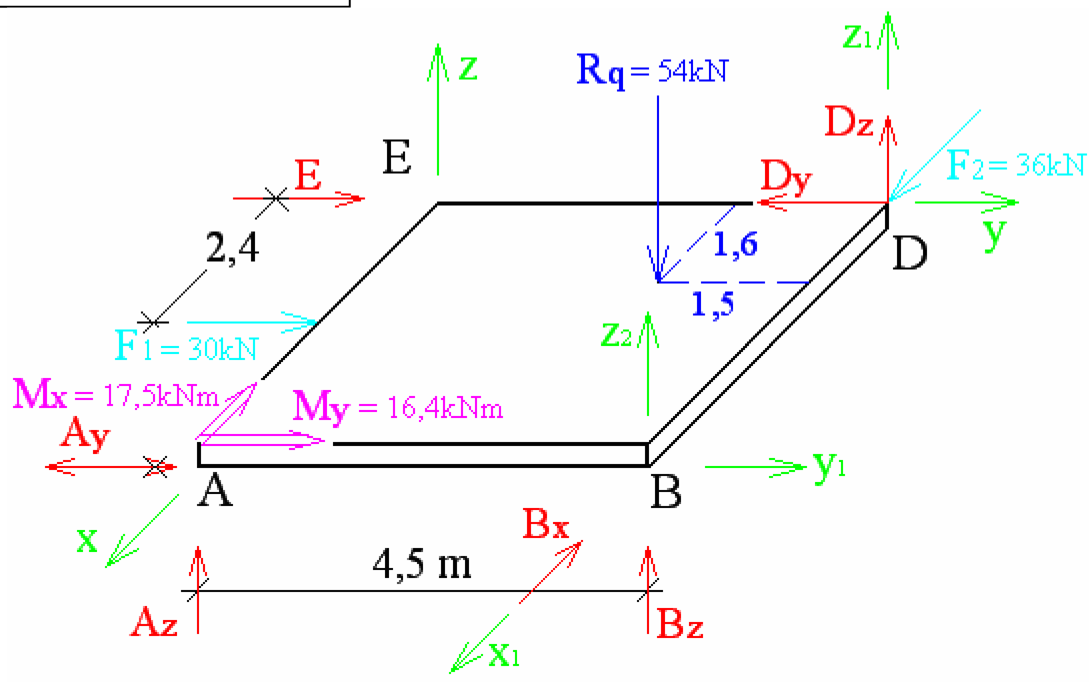


Fig. 5.2.3

3. Check for the reactions obtained

Two force and one moment equations is used for check (Fig.5.2.4):

7) $\sum F_{iy} = 0;\ -A_y + F_1 + E - D_y = 0;\ -15 + 30 + 43,64 - 58,64 = 0;\ 73,64 - 73,64 = 0\,!$

8) $\sum F_{iz} = 0;\ A_z + B_z - R_q + D_z = 0;\ 14,11 + 7,31 - 54 + 32,58 = 0;\ 54 - 54 = 0\,!$

9) $\sum M_{y_2} = 0;\quad M_y - A_z.3,2 - B_z.3,2 + D_z.1,6 = 0;$

$16,4 - 14,11.3,2 - 7,31.3,2 + 32,58.1,6 = 0;$

$68,53 - 68,54 = -0,01 \approx 0\,!$

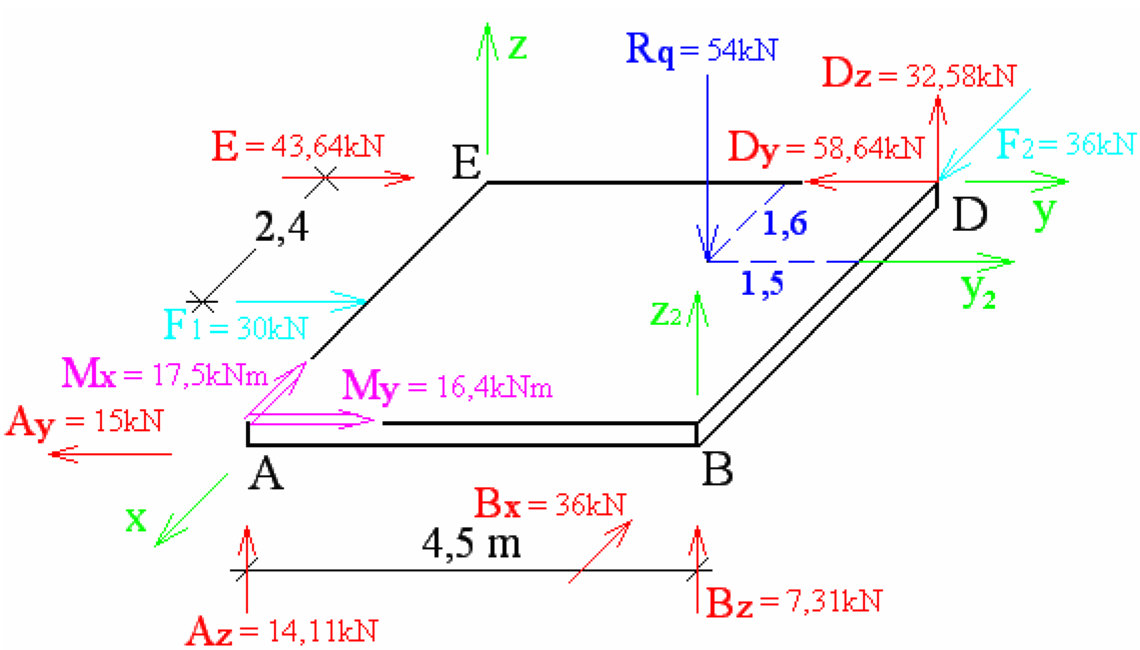


Fig. 5.2.4

COURSE WORK 6: EQUILIBRIUM OF A BODY SUBJECTED TO A SET OF COPLANAR FORCES

Problem 6.1

Apply independent equilibrium equations to determine the support reactions of the construction shown in Fig.6.1.1. Check the result obtained.

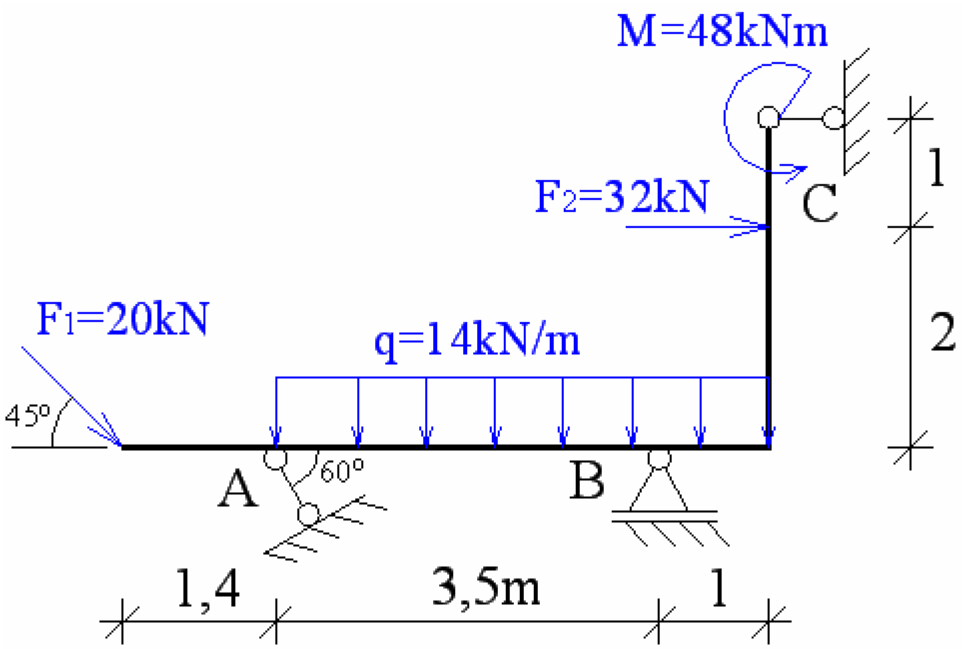


Fig. 6.1.1

Solution:

First step of solution is introduction of the coordinate system of axes x and y (Fig.6.1.2). It should be noted that they play symbolic role here, i.e. they only give horizontal and vertical directions and positive senses. Besides, the origin of the coordinate system is chosen arbitrarily.

1. Determination of the resultant force R_q of the distributed load and the projections of the force F_1 onto axes x and y

- Load q is uniformly distributed and its resultant force is:

$$R_q = q.4{,}5 = 14.4{,}5 = 63\,\text{kN},$$

which direction and sense coincides with the direction and sense of distributed load. The R_q 's point of application is the midpoint of the distance over which the distributed load acts (Fig.6.1.2).

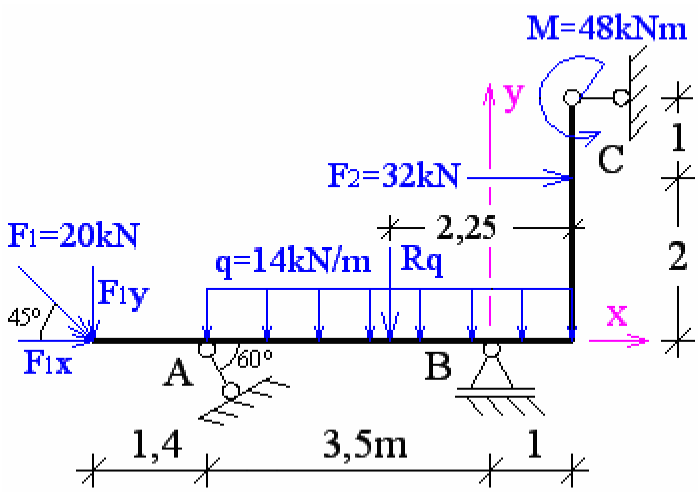


Fig. 6.1.2

• Projections of the force F_1 onto axes x and y are equal, because the angle between the force's line of action and the axes is 45^0, i.e.

$$F_{1x} = F_{1y} = F_1.\cos 45^0 = 20.0{,}707 = 14{,}14\,\text{kN (Fig.6.1.2).}$$

2. Determination of the support reactions

First, the free-body diagram is drawn where the constraints are substituted for their support reactions.

– A is a point where a link inclined at an angle of 60^0 with respect to the horizontal is positioned. Then, a reaction of line of action following direction of the link is introduced. The sense is supposed arbitrarily (Fig.6.1.3);

– Point B is constrained by roller and one vertical support reaction B is introduced instead (Fig.6.1.3);

– There is a horizontal link at point C – the horizontal reaction C is introduced (Fig.6.1.3).

2.1 Projections of reaction A onto axes x and y

The support reaction at point A is in general position with respect to the axes. Therefore, its projections parallel to the axes have to be found. The result is:

$$A_x = A.\cos 60^0 = 0{,}5A;$$

$$A_y = A.\sin 60^0 = 0{,}866A.$$

Their point of application is A, and the senses follow the sense of support reaction A (Fig.6.1.3).

2.2 Independent equilibrium equations

When a coplanar set of forces is applied to the structure, the equations used for the reactions determination are five, two force and three moment equations. Here, the statement requires all equations used for obtaining of support reactions to be independent. Then, the careful analysis of Fig.6.1.3 reveals that the two force equations $\sum F_{ix} = 0$ and $\sum F_{iy} = 0$ are not independent, because each one of them contains two unknowns. It means that the moment equations have to be used. Moreover, these moment equations need to be about points of intersection of the lines of action of two support reactions. These points are named Ritter's points. Thus, intersecting the lines of action of reactions A and B, A and C, B and C, points D, E and H, respectively, are located (Fig.6.1.3).

• **Position of points D and E**

- Right-angled triangle ΔABD is considered (Fig.6.1.3):

$$tg60^0 = \frac{BD}{AB} \Rightarrow BD = AB.tg60^0 = 3{,}5.1{,}732 = 6{,}06\,\text{m};$$

- Right-angled triangle ΔEHD is considered (Fig.6.1.3):

$$tg60^0 = \frac{DH}{EH} \Rightarrow EH = \frac{DH}{tg60^0} = \frac{9{,}06}{1{,}732} = 5{,}23\,\text{m}.$$

2.3 Support reactions determination

The moment equations about Ritter's points D, E, and H are written, as follows. It should be noted that the positive sense of the moment is chosen to be the counterclockwise one (Fig.6.1.3):

1) $\sum M_H = 0;$

$$-A_y.3{,}5 - A_x.3 + F_{1y}.4{,}9 + F_{1x}.3 + R_q.1{,}25 + F_2.1 + M = 0;$$

$$-(0{,}866A).3{,}5 - (0{,}5A).3 + 14{,}14.4{,}9 + 14{,}14.3 + 63.1{,}25 + 32.1 + 48 = 0;$$

$$4{,}531A = 270{,}456;$$

$$A = 59{,}69\,\text{kN}.$$

2) $\sum M_E = 0;$

$$B.5{,}23 - F_{1y}.(5{,}23 - 4{,}9) + F_{1x}.3 - R_q.(5{,}23 - 1{,}25) + F_2.1 + M = 0;$$

$$B.5{,}23 - 14{,}14.0{,}33 + 14{,}14.3 - 63.3{,}98 + 32.1 + 48 = 0;$$

$$5{,}23B = 132{,}99;$$

$$B = 25{,}43\,\text{kN}.$$

3) $\sum M_D = 0;$

$$C.9{,}06 + M - F_2.8{,}06 + R_q.1{,}25 + F_{1y}.4{,}9 - F_{1x}.6{,}06 = 0;$$

$$C.9{,}06 + 48 - 32.8{,}06 + 63.1{,}25 + 14{,}14.4{,}9 - 14{,}14.6{,}06 = 0;$$

$$9{,}06C = 147{,}57;$$

$$C = 16{,}29\,\text{kN}.$$

All of the reactions are obtained positive meaning that the senses are correct (Fig.6.1.3).

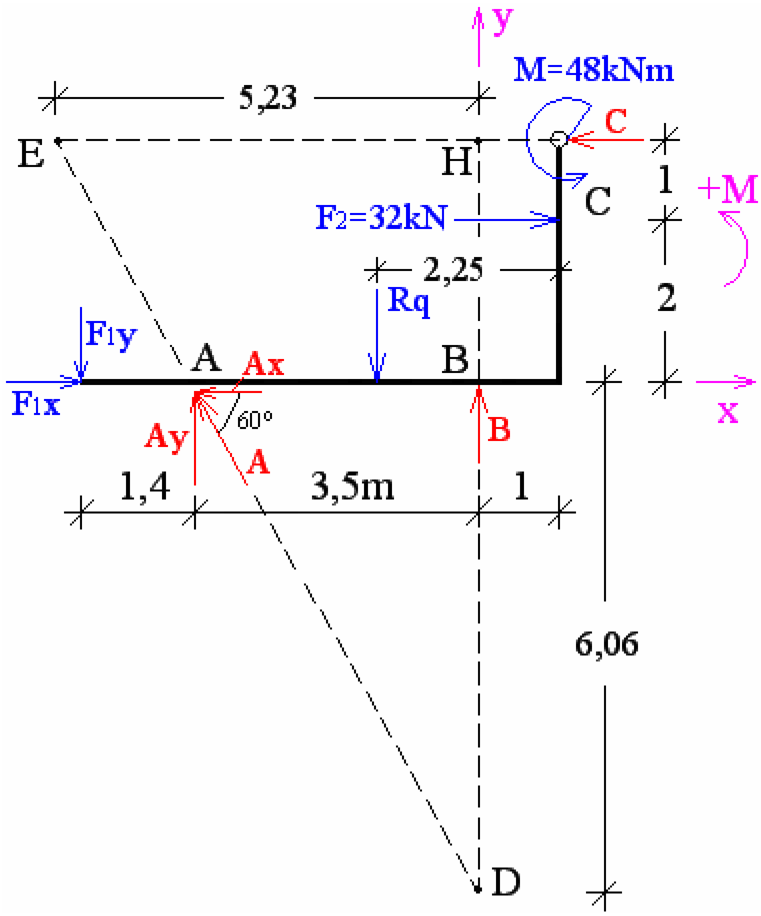


Fig. 6.1.3

3. Check of the results obtained

The two force equations are used for check. First, the magnitudes of A_x and A_y are found:

$$A_x = 0{,}5A = 0{,}5.59{,}69 = 29{,}85\,\text{kN};$$

$$A_y = 0{,}866A = 0{,}866.59{,}69 = 51{,}69\,\text{kN (Fig.6.1.4).}$$

Then:

4) $\sum F_{ix} = 0$; $F_{1x} - A_x + F_2 - C = 0$;

$14{,}14 - 29{,}85 + 32 - 16{,}29 = 0$;

$46{,}14 - 46{,}14 = 0$!

5) $\sum F_{iy} = 0$; $-F_{1y} + A_y - R_q + B = 0$;

$-14{,}14 + 51{,}69 - 63 + 25{,}43 = 0$;

$-77{,}14 + 77{,}12 = -0{,}02 \approx 0$!

The two equations equal zero which means that the reactions obtained are correct!

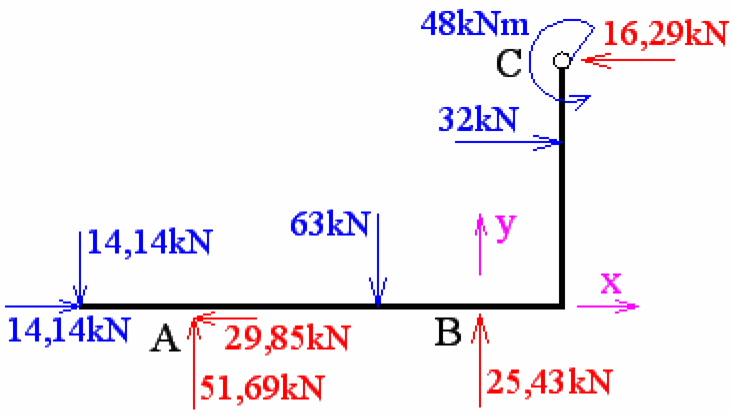


Fig. 6.1.4

<u>Problem 6.2</u>

Apply independent equilibrium equations to determine the support reactions of the construction shown in Fig.6.2.1. Check the result obtained.

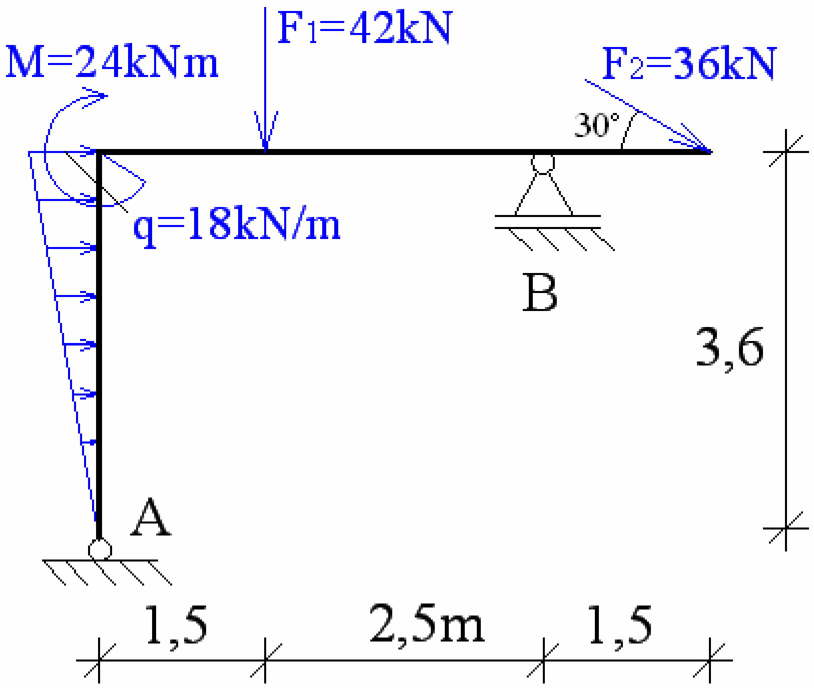


Fig. 6.2.1

Solution:

Solution begins with introduction of the coordinate system of axes x and y (Fig.6.2.2).

1. Determination of R_q, F_{2x} и F_{2y}

• Load q is linearly distributed and its resultant force is:

$$R_q = \frac{1}{2}q.3,6 = \frac{1}{2}18.3,6 = 32,4\,\text{kN};$$

R_q is a force of direction and sense determined by q. The point of application is the projection of the distributed loading's center of gravity on the construction (Fig.6.2.2).

• Projections of F_2 onto axes x and y are:

$$F_{2x} = F_2.\cos 30^0 = 36.0,866 = 31,18\,\text{kN};$$

$$F_{2y} = F_2.\sin 30^0 = 36.0,5 = 18\,\text{kN (Fig.6.2.2).}$$

2. Determination of the support reactions

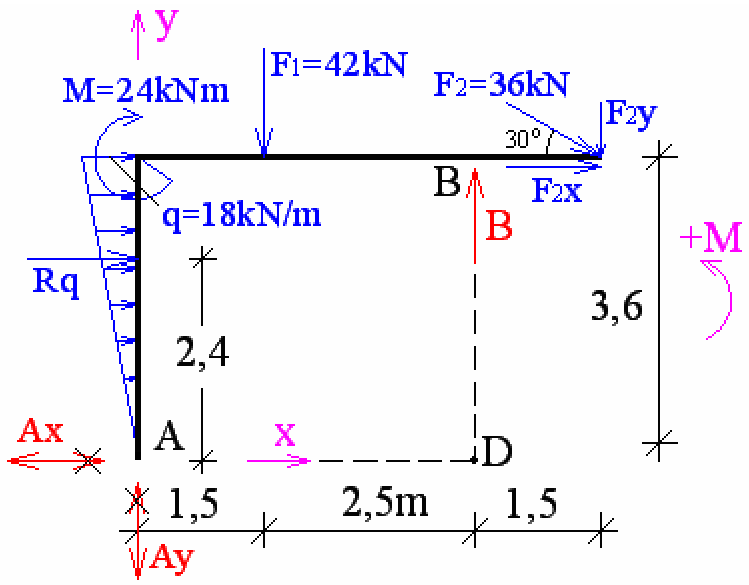


Fig. 6.2.2

The following support reactions are introduced:

– There is a pinned support at point A – a horizontal A_x and a vertical reaction A_y of supposed senses are introduced (Fig.6.2.2);

– The roller is situated at point B – a vertical support reaction B is introduced (Fig.6.2.2).

Analyzing Fig.6.2.2, it is concluded that only one unknown reaction is parallel to *x*-axis. Then, the first independent equation is:

$$1)\ \sum F_{ix} = 0;\ A_x + R_q + F_{2x} = 0;$$
$$A_x + 32{,}4 + 31{,}18 = 0;$$
$$A_x = -63{,}58\,\text{kN}.$$

The negative sign means that the sense chosen for A_x is not correct and has to be changed (Fig.6.2.2).

The second independent equilibrium equation is the moment one about point A, because A is the point of intersection of the A_x and A_y's lines of action (Fig.6.2.2):

$$2)\ \sum M_A = 0; \qquad -R_q.2{,}4 - M - F_1.1{,}5 + B.4 - F_{2x}.3{,}6 - F_{2y}.5{,}5 = 0;$$
$$-32{,}4.2{,}4 - 24 - 42.1{,}5 + B.4 - 31{,}18.3{,}6 - 18.5.5 = 0;$$
$$4B = 376;$$
$$B = 94\,\text{kN}.$$

The positive sense of moment is the counterclockwise one.

The third independent equation is the moment equation about D, the point of intersection A_x and B's lines of action (Fig.6.2.2):

$$3)\ \sum M_D = 0; \qquad -A_y.4 - R_q.2{,}4 - M + F_1.2{,}5 - F_{2x}.3{,}6 - F_{2y}.1{,}5 = 0;$$
$$-A_y.4 - 32{,}4.2{,}4 - 24 + 42.2{,}5 - 31{,}18.3{,}6 - 18.1{,}5 = 0;$$
$$4A_y = -136;$$
$$A_y = -34\,\text{kN}.$$

3. Check for the reactions obtained

Since one force and two moment equations have been used for the support reaction determination, a moment equation containing all reactions is used for check. This equation is the moment one about point E (Fig. 6.2.3):

$$4)\ \sum M_E = 0; \qquad A_y.1{,}5 - A_x.2{,}4 - M + B.2{,}5 - F_{2x}.1{,}2 - F_{2y}.4 = 0;$$
$$34.1{,}5 - 63{,}58.2{,}4 - 24 + 94.2{,}5 - 31{,}18.1{,}2 - 18.4 = 0;$$
$$286 - 286 = 0\,!$$

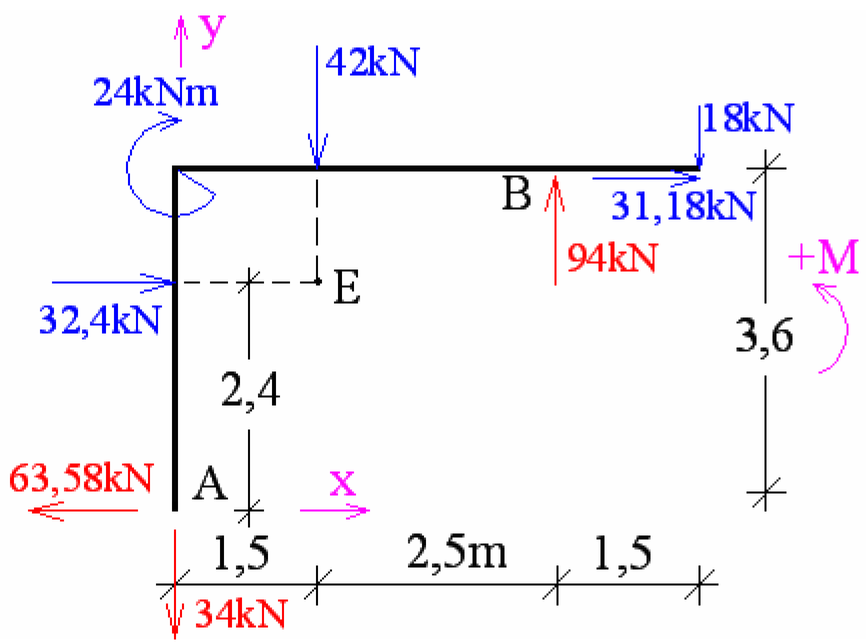


Fig. 6.2.3

COURSE WORKS 7: EQUILIBRIUM OF GERBER BEAMS

Problem 7.1

Determine the support reactions and the forces in hinges of the Gerber beam shown in Fig.7.1.1. Check the result obtained.

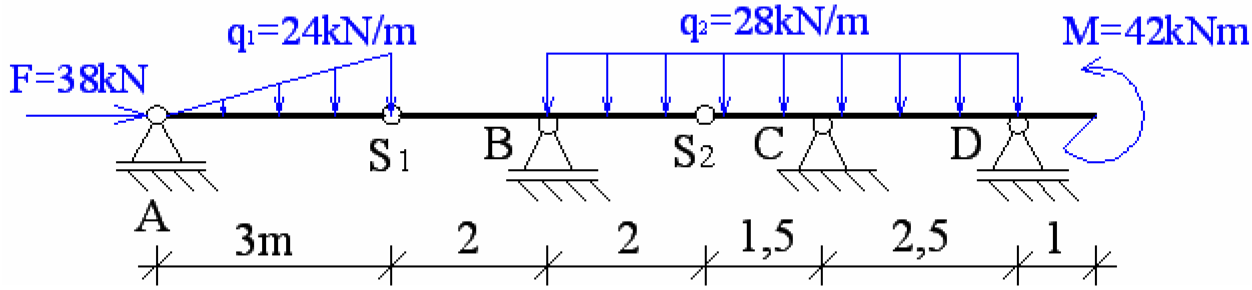


Fig. 7.1.1

Solution:

First, the number and the type of the beams composing the Gerber beam are determined. Here, the beams are three, namely AS_1, S_1BS_2, and S_2CD. Beam S_2CD is the main beam, because it will be in equilibrium detached from the structure. However, beams AS_1 and S_1BS_2 can not be in equilibrium without the main beam. Therefore, they are secondary beams. Then, the beams are numbered: AS_1 is beam (1), S_1BS_2 is beam (2), and S_2CD is beam (3). Finally, the coordinate system of axes x and y is introduced (Fig.7.1.2).

1. Determination of the distributed loads' resultant forces

• Load q_1 is applied on beam (1). It is linearly distributed and its resultant force is:

$$R_1 = \frac{1}{2} q_1.\overline{AS_1} = \frac{1}{2} 24.3 = 36 \text{ kN}.$$

R_1 has line of action and sense following these ones of distributed load. The point of application of R_1 is the position of the center of gravity of the triangle representing the distributed load (Fig.7.1.2).

• The load q_2 is uniformly distributed over two beams. Then, two resultant forces have to be obtained.

- Resultant force for the portion of q_2 applied on beam (2) is:

$$R_2 = q_2.\overline{BS_2} = 28.2 = 56 \text{ kN}.$$

R_2 is a vertical force of downward sense with point of application at the midpoint of distance $\overline{BS_2}$ (Fig.7.1.2).

- Resultant force for the portion of q_2 acting on beam (3) is:

$$R_3 = q_2.\overline{DS_2} = 28.4 = 112 \text{ kN}.$$

R_3 is a vertical force of downward sense with point of application at the midpoint of distance $\overline{DS_2}$ (Fig.7.1.2).

2. Determination of the support reactions and the forces in hinges

In order to determine the support reactions and forces in hinges, the Gerber beam is separated through the hinges. In other words, the entire construction is decomposed into three beams over which the applied loads and unknown reactions and forces in hinges are introduced. The number and the lines of action of the support reactions depends on the type of the constraints, and, as usual, their senses are arbitrary chosen (Fig.7.1.2). The forces in hinges are introduced at points S_1 and S_2, and they act on the two beams connected by the hinge. By definition, the forces in hinges represent the influence of one beam to another and they ensure the equilibrium of the beam detached from entire structure. The forces in hinges have vertical and horizontal lines of action and senses supposed randomly, but opposite for the two beams connected by the hinge (Fig.7.1.2). Finally, it should be noted that the solution starts from the beam of fewest number unknowns. Here, this is beam (1).

2.1 Beam (1)

It is convenient the three independent equations to be written. These are:

1) $\sum F_{ix} = 0$; $F - S_{1x} = 0$; $38 - S_{1x} = 0$; $S_{1x} = 38 \text{ kN}$;

2) $\sum M_A = 0$; $-R_1.2 + S_{1y}.3 = 0$; $-36.2 + S_{1y}.3 = 0$; $S_{1y} = 24 \text{ kN}$.

The positive sense of the moment is chosen to be the counterclockwise one (Fig.7.1.2).

3) $\sum M_{S_1} = 0;\ -R_1.1 + A.3 = 0; -36.2 + A.3 = 0;\ A = 12\,\text{kN}.$

The three unknowns are obtained positive meaning that the senses chosen are correct (Fig.7.1.2).

- Check of the result obtained: $\sum F_{iy} = 0;\ A - R_1 + S_{1y} = 0;\ 12 - 36 + 24 = 0;\ 36 - 36 = 0\,!$

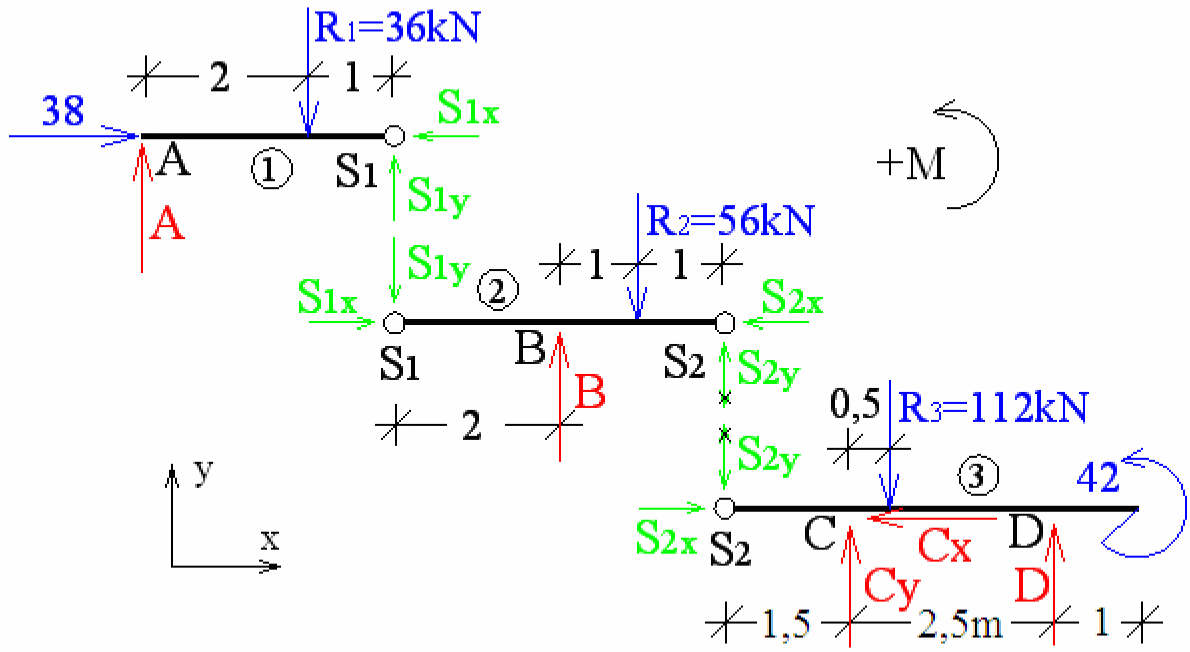


Fig. 7.1.2

2.2 Beam (2)

1) $\sum F_{ix} = 0;\ S_{1x} - S_{2x} = 0;\ 38 - S_{2x} = 0;\ S_{2x} = 38\,\text{kN};$

2) $\sum M_B = 0;\ S_{1y}.2 - R_2.1 - S_{2y}.2 = 0; 24.2 - 56.1 - S_{2y}.2 = 0;\ S_{2y} = -4\,\text{kN};$

S_{2y} is obtained negative, i.e. its sense is changed for the two beams to which S_{2y} is applied (Fig.7.1.2).

3) $\sum M_{S_2} = 0;\ S_{1y}.4 - B.2 + R_2.1 = 0; 24.4 - B.2 + 56.1 = 0;\ B = 76\,\text{kN}.$

- Check: $\sum F_{iy} = 0;\ -S_{1y} + B - R_2 + S_{2y} = 0;\ -24 + 76 - 56 + 4 = 0;\ 80 - 80 = 0\,!$

2.3 Beam (3)

1) $\sum F_{ix} = 0;\ S_{2x} - C_x = 0;\ 38 - C_x = 0;\ C_x = 38\,\text{kN};$

2) $\sum M_C = 0;\ S_{2y}.1{,}5 - R_3.0{,}5 + D.2{,}5 + M = 0; 4.1{,}5 - 112.0{,}5 + D.2{,}5 + 42 = 0;\ D = 3{,}2\,\text{kN};$

3) $\sum M_D = 0;\ S_{2y}.4 - C_y.2{,}5 + R_3.2 + M = 0; 4.4 - C_y.2{,}5 + 112.2 + 42 = 0;\ C_y = 112{,}8\,\text{kN}.$

- Check: $\sum F_{iy} = 0;\ -S_{2y} + C_y - R_3 + D = 0;\ -4 + 112{,}8 - 112 + 3{,}2 = 0;\ 116 - 116 = 0\,!$

3. Check of the entire structure

In order to perform this check, the Gerber beam is recomposed and the support reactions obtained are applied to it. The equations for check are the following ones:

1) $\sum F_{ix} = 0;\ F_1 - C_x = 0;\ 38 - 38 = 0\,!$

2) $\sum F_{iy} = 0;\ A - R_1 + B - R_2 + C_y - R_3 + D = 12 - 36 + 76 - 56 + 112{,}8 - 112 + 3{,}2 = 0;\ 204 - 204 = 0\,!$

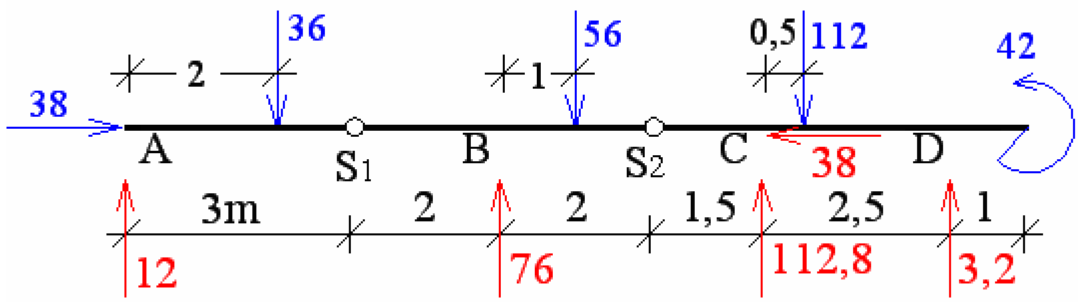


Fig. 7.1.3

COURSE WORK 8: EQUILIBRIUM OF THREE-HINGED FRAMES

Problem 8.1

Determine the support reactions and the forces in hinge of the three-hinged frame shown in Fig.8.1.1. Check the result obtained.

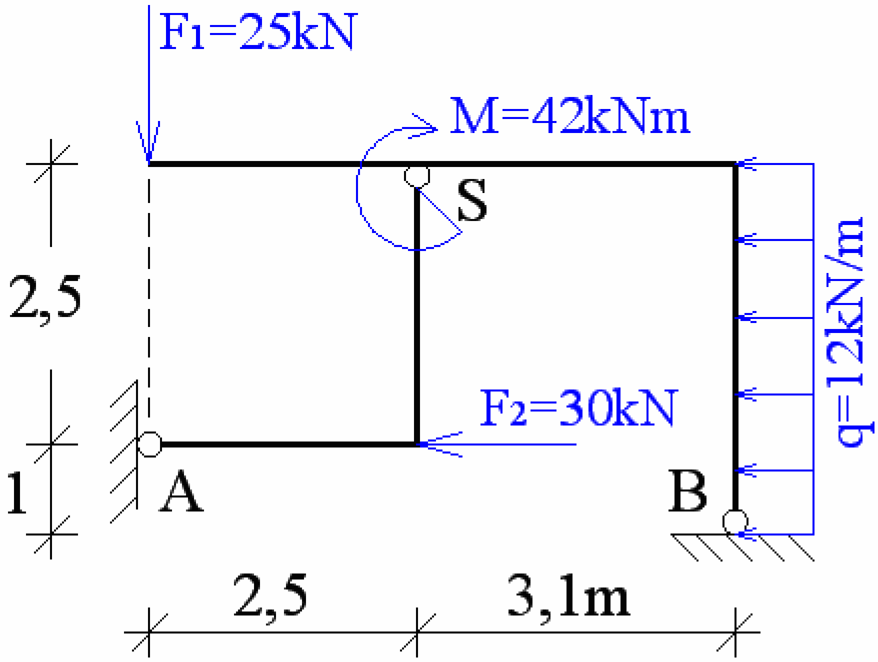


Fig. 8.1.1

Solution:

1. Determination of the resultant force of distributed load

The load q is uniformly distributed and its resultant force is:

$$R_q = 12.3{,}5 = 42\,\text{kN}.$$

R_q is a horizontal force of sense to the left. The point of application of R_q is the midpoint of the distance over which the load is distributed (Fig.8.1.2).

2. Determination of the support reactions

First, free-body diagram of entire structure is drawn. Points A and B are pinned supports and they are substituted for their reactions having horizontal and vertical lines of action and senses chosen arbitrarily (Fig.8.1.2a). However, the number of the unknown for entire three-hinged frame is four while the number of equilibrium equations is three. Then, in order to find the unknowns, the three-hinged frame is separated to its composing two parts through the hinge. Further, each part is loaded by the external forces and support reactions acting upon it. In the case when a concentrated force or moment is applied directly at the hinge, then, the force or moment is assumed to act only on one part of the construction. Here, it is chosen the moment to be applied to the frame part AS (Fig.8.1.2b). After that, in order to ensure the equilibrium of each part of the frame, the forces in hinges are introduced at point S. It should be noted that their lines of action are horizontal and vertical while their senses are arbitrary chosen. Besides, they are applied to the two parts of the frame and their senses are opposite with regard to the two parts (Fig.8.1.2b).

General type of three-hinged frame does not allow the independent equilibrium equations to be written. Then, a set of two equations of two unknowns has to be found. The proper choice is first equation of the set to be the moment equation for entire structure, while the second equation to be the moment equation for one of the parts. The points, about which the moment equations are going to be written, are the points of intersection of two unknowns.

- **Determination of** A_x **and** A_y

1) $\sum M_B^{Ent.Str.} = 0$ (Fig.8.1.2a);

$-A_y.5{,}6 - A_x.1 + F_2.1 + F_1.5{,}6 - M + R_q.1{,}75 = 0$;

$-A_y.5{,}6 - A_x.1 + 30.1 + 25.5{,}6 - 42 + 42.1{,}75 = 0$;

$A_x + 5{,}6A_y = 201{,}5$.

The positive sense of the moment is chosen to be the counterclockwise one.

2) $\sum M_S^{AS} = 0$ (Fig.8.1.2b);

$-A_y.2{,}5 + A_x.2{,}5 - F_2.2{,}5 - M = 0$;

$-A_y.2{,}5 + A_x.2{,}5 - 30.2{,}5 - 42 = 0$;

$A_x - A_y = 46{,}8$.

Solving the set of equations, the result is:

$A_x + 5{,}6A_y = 201{,}5$ (1)

$A_x - A_y = 46{,}8$ (2)

$A_x - A_y = 46{,}8 \Rightarrow A_x = 46{,}8 + A_y$

$46{,}8 + A_y + 5{,}6A_y = 201{,}5$

$6{,}6A_y = 154{,}7$

$A_y = 23{,}44$ kN;

$A_x = 46{,}8 + 23{,}44 = 70{,}24$ kN.

A_x and A_y have positive values, i.e. their senses are correct.

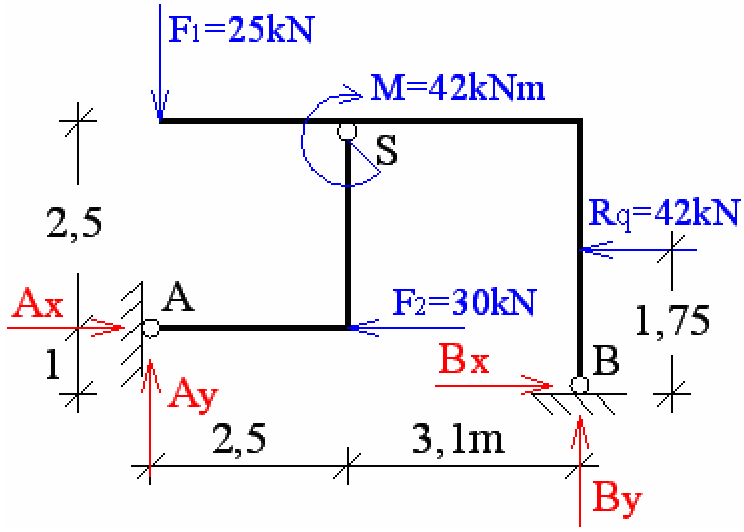


Fig. 8.1.2a

- **Determination of B_x and B_y**

3) $\sum M_A^{Ent.Str.} = 0$ (Fig.8.1.2a);

$-M + R_q.0{,}75 + B_x.1 + B_y.5{,}6 = 0$;

$-42 + 42.0{,}75 + B_x.1 + B_y.5{,}6 = 0$;

$B_x + 5{,}6B_y = 10{,}5$.

4) $\sum M_S^{BS} = 0$ (Fig.8.1.2b);

$F_1.2{,}5 - R_q.1{,}75 + B_x.3{,}5 + B_y.3{,}1 = 0$;

$25.2{,}5 - 42.1{,}75 + B_x.3{,}5 + B_y.3{,}1 = 0$;

$3{,}5B_x + 3{,}1B_y = 11$.

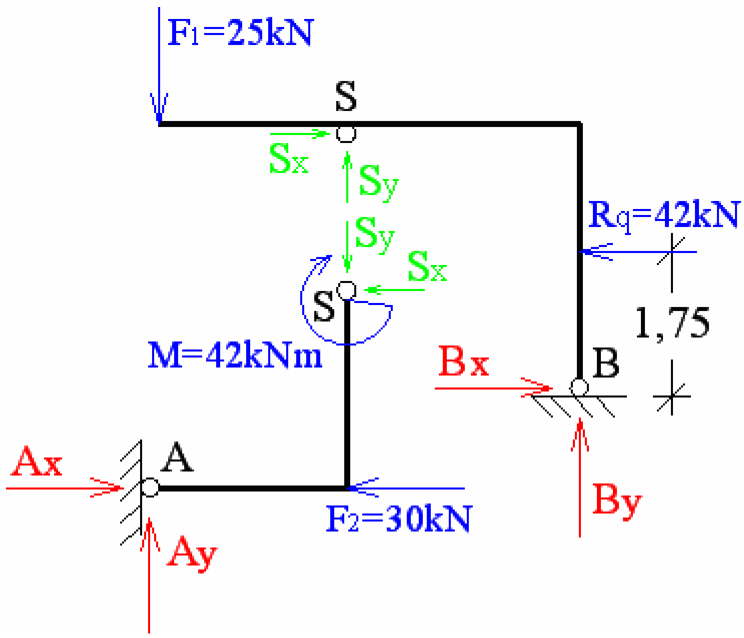


Fig. 8.1.2b

Solving the set of equations, it is obtained:

$B_x + 5{,}6B_y = 10{,}5$ (3)

$3{,}5B_x + 3{,}1B_y = 11$ (4)

$B_x + 5{,}6B_y = 10{,}5 \Rightarrow B_x = 10{,}5 - 5{,}6B_y$

$3{,}5(10{,}5 - 5{,}6B_y) + 3{,}1B_y = 11$

$36{,}75 - 19{,}6B_y + 3{,}1B_y = 11$

$-16{,}5B_y = -25{,}75$

$B_y = 1{,}56$ kN;

$B_x = 10{,}5 - 5{,}6.1{,}56 = 1{,}76$ kN.

- **Check for the reactions obtained** (Fig.8.1.3):

$\sum F_{ix}^{Ent.Str.} = 0$; $A_x - F_2 - R_q + B_x = 0$;
$70{,}24 - 30 - 42 + 1{,}76 = 0$;
$72 - 72 = 0$!

$\sum F_{iy}^{Ent.Str.} = 0$; $A_y - F_1 + B_y = 0$;
$23{,}44 - 25 + 1{,}56 = 0$;
$25 - 25 = 0$.

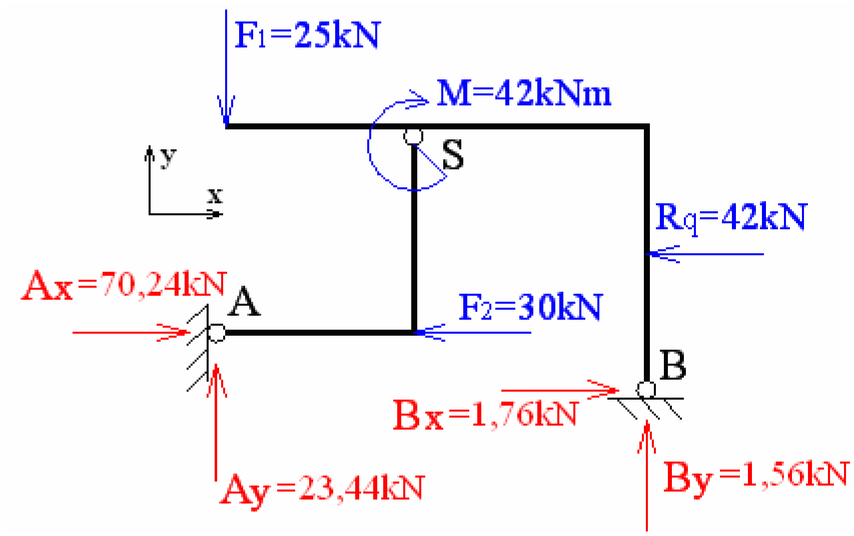


Fig. 8.1.3

3. Determination of the forces in hinge

The forces in hinge are determined by two force equation for one of the frame parts. Then, they are checked by two force equation for the other part (Fig.8.1.4):

$\sum F_{ix}^{AS} = 0$; $A_x - F_2 - S_x = 0$; $70{,}24 - 30 - S_x = 0$; $S_x = 40{,}24$ kN.

$\sum F_{iy}^{AS} = 0$; $A_y - S_y = 0$; $23{,}44 - S_y = 0$; $S_y = 23{,}44$ kN.

- Check:

$\sum F_{ix}^{BS} = 0$; $S_x - R_q + B_x = 0$; $40{,}24 - 42 + 1{,}76 = 0$; $42 - 42 = 0$!

$\sum F_{iy}^{BS} = 0$; $S_y - F_1 + B_y = 0$; $23{,}44 - 25 + 1{,}56 = 0$; $25 - 25 = 0$!

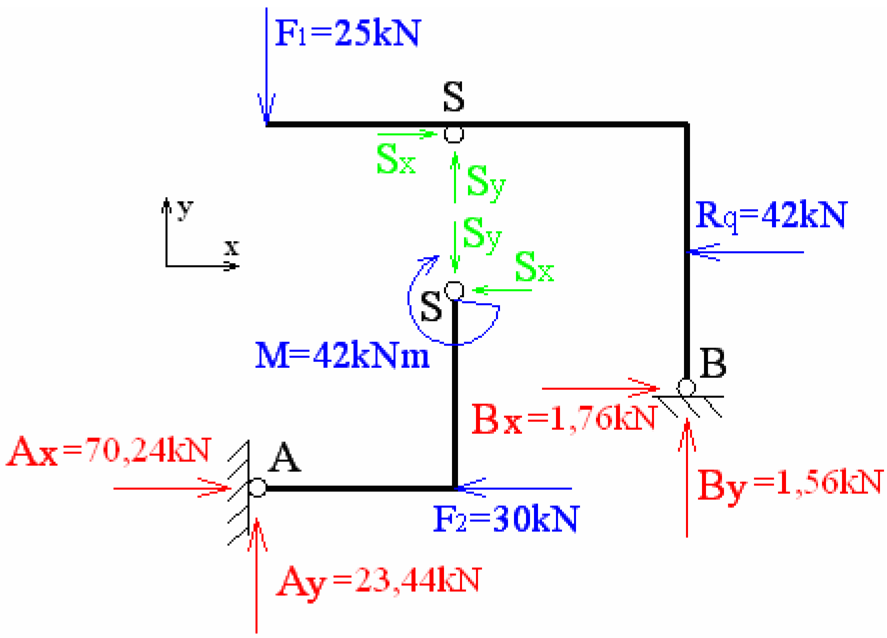


Fig. 8.1.4

COURSE WORK 9: EQUILIBRIUM OF PLANE TRUSSES

Problem 9.1

A plane truss is supported and loaded as shown in Fig.9.1.1. Determine:

1. Support reactions;
2. Force in each member using the method of joints;
3. Force in the members marked using the method of sections.

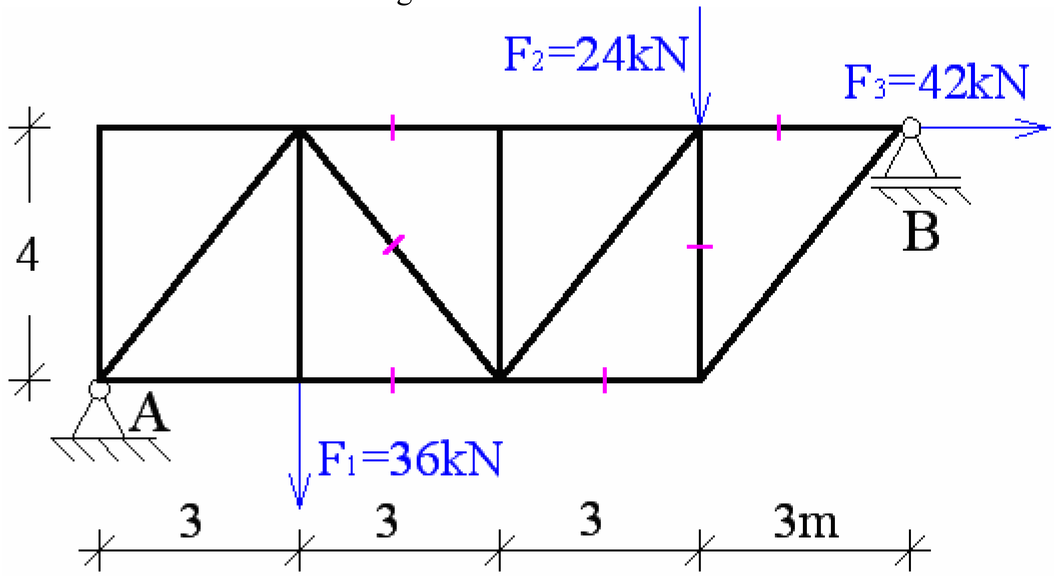


Fig. 9.1.1

Solution:

1. Support reactions determination

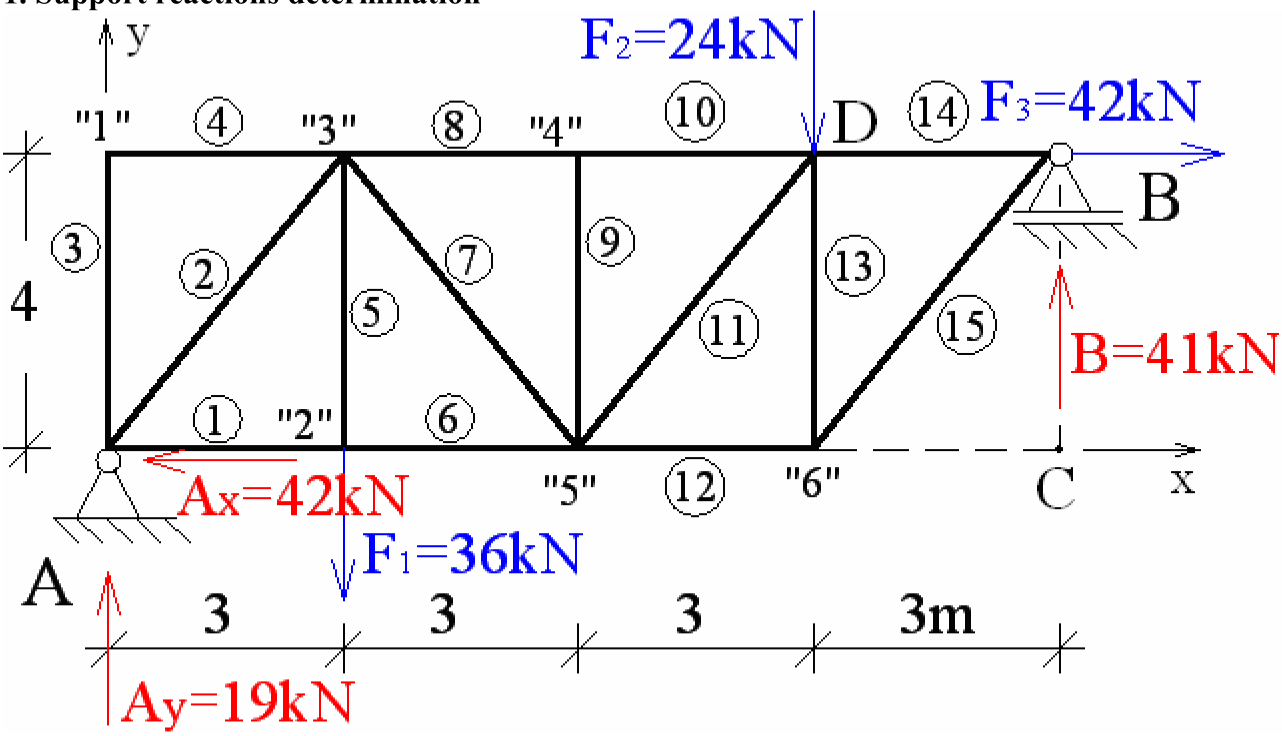


Fig. 9.1.2

Here, the independent equations for determination of the support reactions are one force equation onto x-axis and two moment equations about the point of intersection of A_x and A_y (point A) and the point of intersection of A_x and B (point C), in which the positive sense of the moment is assumed to be the counterclockwise one (Fig.9.1.2):

1) $\sum F_{ix} = 0 \;\Rightarrow\; -A_x + F_3 = 0 \Rightarrow -A_x + 42 = 0 \Rightarrow A_x = 42$ kN;

2) $\sum M_A = 0 \;\Rightarrow -F_1.3 - F_2.9 - F_3.4 + B.12 = 0 \Rightarrow -36.3 - 24.9 - 42.4 + B.12 = 0 \Rightarrow B = 41$ kN;

3) $\sum M_C = 0 \;\Rightarrow -A_y.12 + F_1.9 + F_2.3 - F_3.4 = 0 \Rightarrow -A_y.12 + 36.9 + 24.3 - 42.4 = 0 \Rightarrow A_y = 19$ kN.

- Check: $\sum M_D = 0 \;\Rightarrow -A_y.9 - A_x.4 + F_1.6 + B.3 = 0; -19.9 - 42.4 + 36.6 + 41.3 = 0; 339 - 339 = 0\,!$

2. Determination of the force in each member using the method of joints

2.1 Numeration of links and joints

Numeration begins from the left and goes to the right. The numbers of the members are in circle, while the numbers of the joints are in quotation marks (Fig.9.1.2).

2.2 Zero-force members

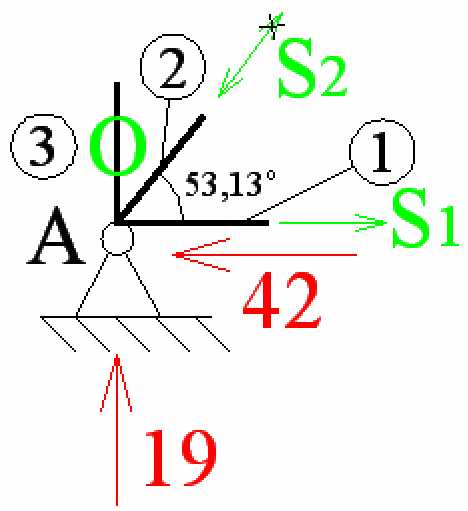
In present problem, the zero-force members are three:

- links (3) and (4) – they are connected by the unloaded joint „1" (Fig.9.1.2);
- link (9) – it ends in joint „4" where the other two members have one and the same direction (Fig.9.1.2).

2.3 Forces in the links

Solution begins from the first joint, here, joint A. It is detached from the truss and the external forces and the forces in members are introduced. The forces in the truss members are unknown – their lines of action coincide with the directions of the links while their senses are chosen to be out of the cut, i.e. the forces are supposed tensile. Then, the equilibrium of the joint is examined using two force equations. Thus, the unknowns are obtained, as follows.

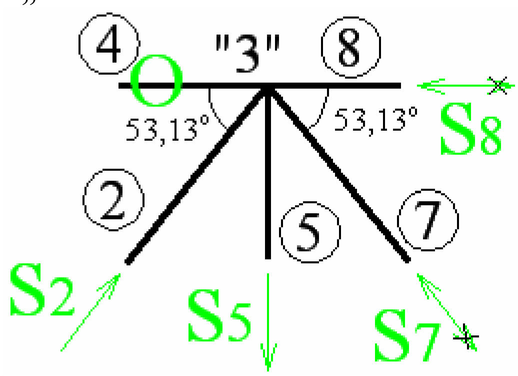
- Joint „A"



$\sum F_{iy} = 0; S_2 \sin 53{,}13^0 + 19 = 0;\; S_2 = -23{,}75$ kN (c);

$\sum F_{ix} = 0; S_1 - 42 - S_2 \cos 53{,}13^0 = 0; S_1 = 56{,}25$ kN(t).

- Joint „3"



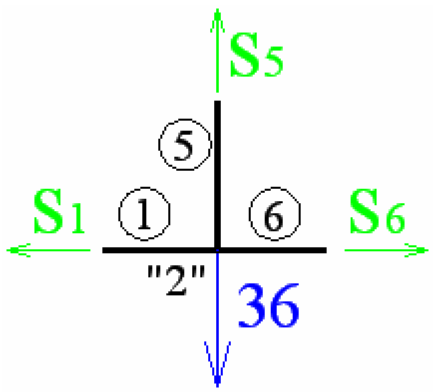
$\sum F_{iy} = 0; S_2 \sin 53{,}13 - S_5 - S_7 \sin 53{,}13 = 0;$

$S_7 = -21{,}25$ kN (c);

$\sum F_{ix} = 0; S_2 \cos 53{,}13 + S_8 - S_7 \cos 53{,}13 = 0;$

$S_8 = -1{,}5$ kN (t).

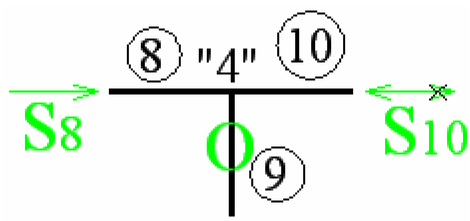
- Joint „2"



$\sum F_{ix} = 0 \Rightarrow S_6 - S_1 = 0;\; S_6 = 56{,}25$ kN (t);

$\sum F_{iy} = 0 \Rightarrow S_5 - 36 = 0;\; S_5 = 36$ kN (t).

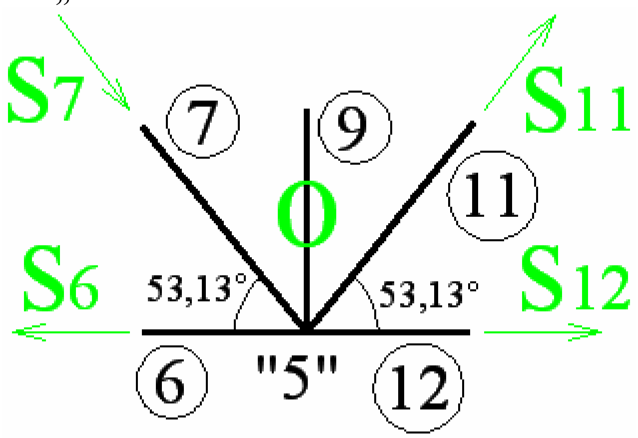
- Joint „4"



$\sum F_{ix} = 0 \Rightarrow -S_8 - S_{10} = 0;\; S_{10} = -1{,}5$ kN (c);

$\sum F_{iy} = 0 \Rightarrow S_9 = 0$.

- Joint „5"



$\sum F_{iy} = 0\,;\, S_7 \sin 53{,}13 - S_{11} \sin 53{,}13 = 0\,;$

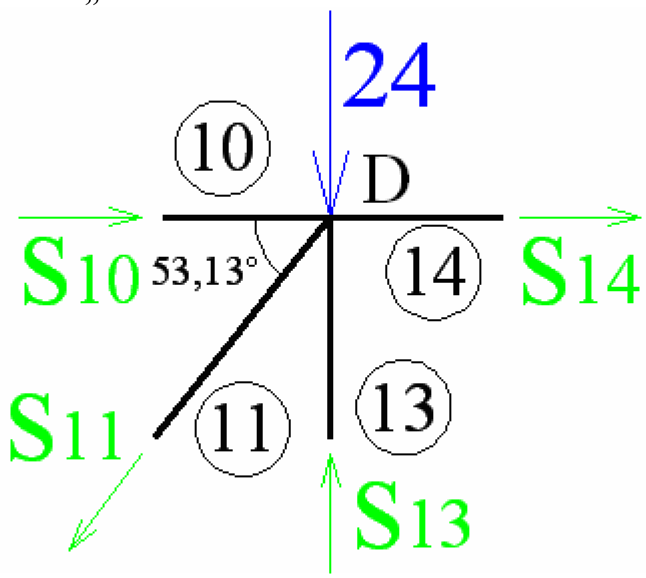
$S_{11} = 21{,}25 \text{ kN (t)};$

$\sum F_{ix} = 0\,;$

$- S_6 + S_7 \cos 53{,}13 + S_{11} \cos 53{,}13 + S_{12} = 0\,;$

$S_{12} = 30{,}75 \text{ kN (t)}.$

- Joint „D"



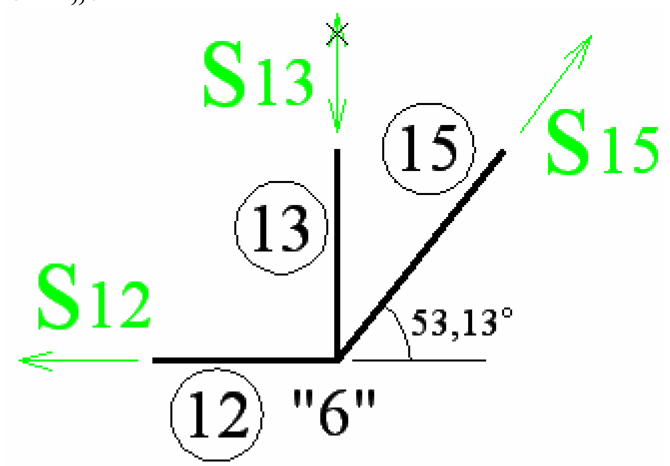
$\sum F_{ix} = 0\,;\, S_{10} - S_{11} \cos 53{,}13 + S_{14} = 0\,;$

$S_{14} = 11{,}25 \text{ kN (t)}.$

$\sum F_{iy} = 0\,;\; -24 - S_{11} \sin 53{,}13 + S_{13} = 0\,;$

$-24 - 21{,}25.0{,}8 + 41 = 0\,;\; 41 - 41 = 0\,!$

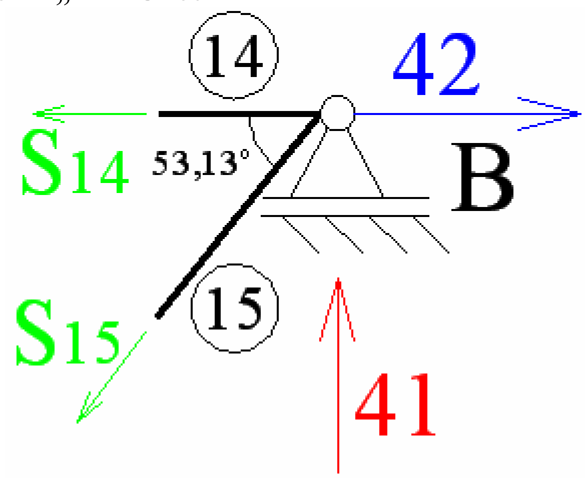
- Joint „6"



$\sum F_{ix} = 0\,; - S_{12} + S_{15} \cos 53{,}13^0 = 0\,; S_{15} = 51{,}25 \text{ kN(t)};$

$\sum F_{ix} = 0\,;\; S_{13} + S_{15} \sin 53{,}13^0 = 0\,; S_{13} = -41 \text{ kN(c)}.$

- Joint „B" - Check



$\sum F_{ix} = 0\,; - S_{14} - S_{15} \cos 53{,}13 + 42 = 0\,;$

$-11{,}25 - 51{,}25.0{,}6 + 42 = 0\,;\; 42 - 42 = 0\,!$

$\sum F_{iy} = 0\,;\; 41 - S_{15} \sin 53{,}13 = 0\,;$

$41 - 51{,}25.0{,}8 = 0\,;\; 41 - 41 = 0\,!$

Notes:

1. There are letters (c) or (t) behind the values obtained for each force in truss members. The letters shows that the rod is compressive or tensile, respectively. When the rod is compressed, the sense of the force has to be changed!

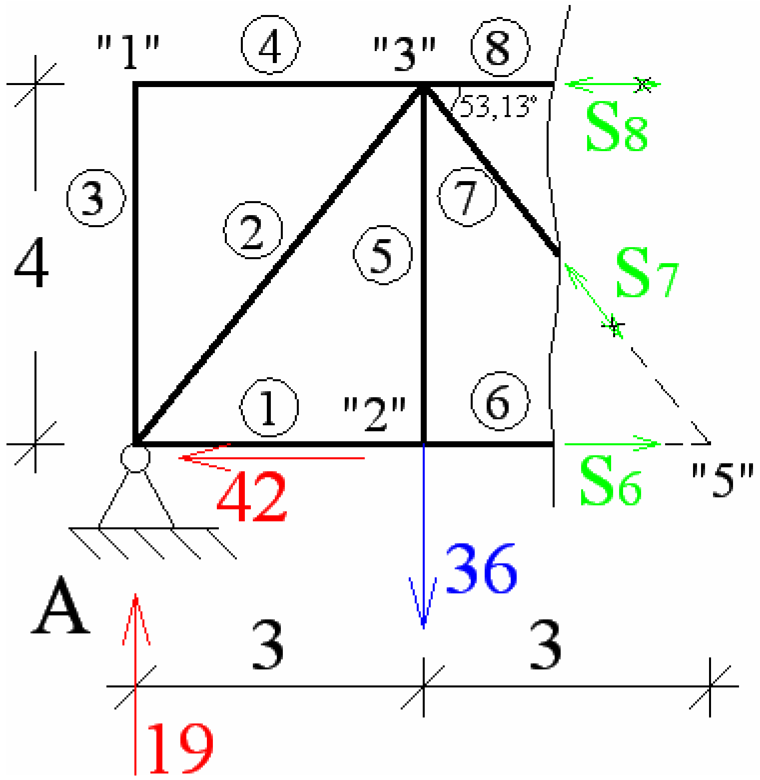
2. The last joint's equations of equilibrium are used for the check of the results obtained.

3. Method of joints allows solution of the problem to begin from the last joint and to go from the right to the left. Then, joint A will be used for the check of the results.

3. Determination of the force in truss members marked using the method of sections

The essence of the method of sections is that the truss is divided into two portions by an imaginary cut through the three members marked. Then, the truss portion loaded by less number of loads is considered where the external forces, support reactions and the forces in the members through which the cut passes are applied. Further, three independent equilibrium equations are written and the unknowns are obtained. Finally, the check is made.

3.1 Forces in members 6, 7, and 8



1) $\sum F_{iy} = 0; 19 - 36 - S_7 \sin 53{,}13^0 = 0;$ $S_7 = -21{,}25\,\text{kN (c)};$

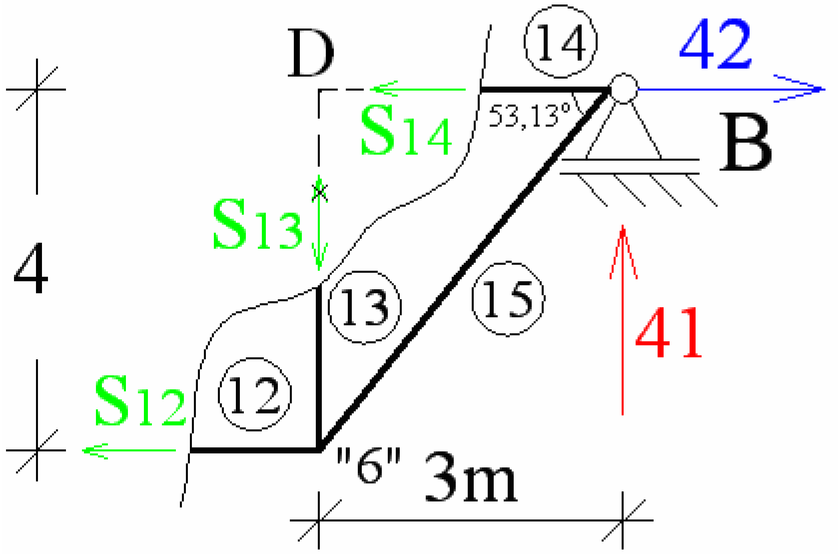
2) $\sum M_{"5"} = 0;\ -19.6 + 36.3 - S_8.4 = 0;$ $S_8 = -1{,}5\,\text{kN (c)};$

3) $\sum M_{"3"} = 0;\ -42.4 - 19.3 + S_6.4 = 0;$ $S_6 = 56{,}25\,\text{kN (t)};$

- Check

4) $\sum F_{ix} = 0;\ -42 - S_8 + S_6 - S_7 \cos 53{,}13^0 = 0;\ -42 - 1{,}5 + 56{,}25 - 21{,}25.0{,}6 = 0;\ 56{,}25 - 56{,}25 = 0\,!$

3.2 Forces in members 12, 13, and 14



1) $\sum F_{iy} = 0; S_{13} + 41 = 0;$ $S_{13} = -41\,\text{kN (c)};$

2) $\sum M_D = 0;\ 41.3 - S_{12}.4 = 0;$ $S_{12} = 30{,}75\,\text{kN (t)};$

3) $\sum M_{"6"} = 0;\ S_{14}.4 - 42.4 + 41.3 = 0;$ $S_{14} = 11{,}25\,\text{kN (t)};$

- Check: $\sum F_{ix} = 0;$ $-S_{12} - S_{14} + 42 = 0;\ -30{,}75 - 11{,}25 + 42 = 0;\ 42 - 42 = 0\,!$