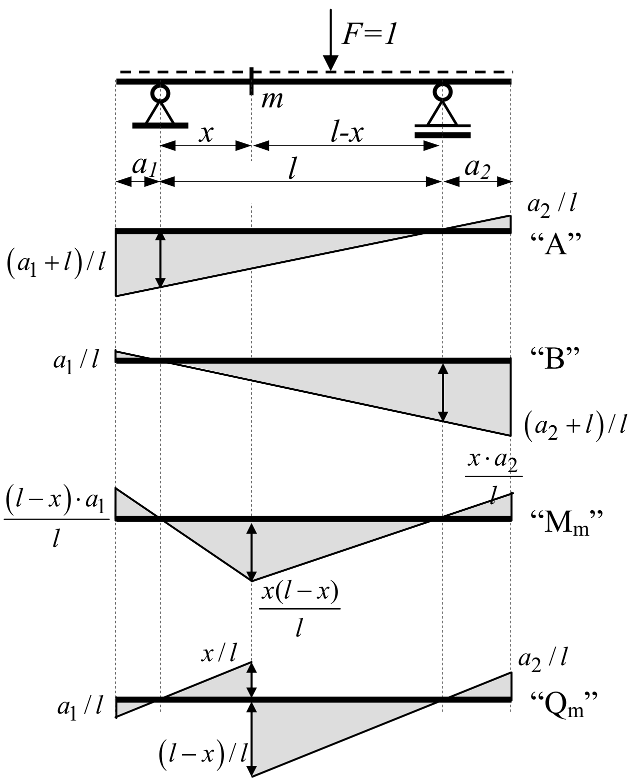
Influence lines in trusses

I. Basic concepts.

The maximum design force in a member of a structure subjected to a system of stationary loads is completely determined. However, a member such as bridge girder or a crane gantry girder are subjected to moving loads, and the maximum design force in such members depends on the location of the load. The determination of the location of the moving load that produces the maximum design force in a member is reached by using influence lines.
Definition: *An influence line for a member is a graph representing the variation of internal force in a fixed section of the member, due to a unit load traversing a structure.*

Influence lines in a simply supported beam



The ordinates of the influence line for the reaction "A" represent the value of this reaction at the instant when the F=1 is placed directly over this cross section.

Any ordinate for the influence line "M_m" depicts the value of the bending moment in section *m* when the unit load is placed over this particular ordinate.

Ordinates of the shear force influence line represent the shear force in the section *m* arising from a unit load acting in the section corresponding to the said ordinate.

II. Influence lines in trusses

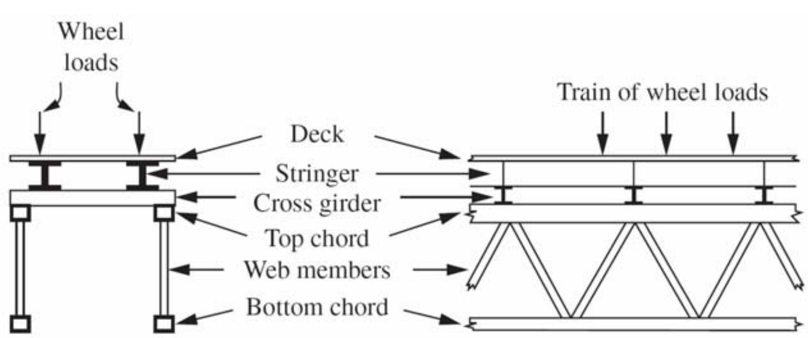


Figure 1 Loads transfer by system of stringers and cross beams

As moving loads traverse a pin jointed truss, the loads are transferred to the truss panel nodes by a system of stringers and cross beams (Fig. 1). The moving load is transferred from one panel

point to the next as the load moves across the stringer. Hence, the influence line for axial force in a member is completed by connecting the influence line ordinates at the panel points on either side of a panel with a straight line.
All the methods used for computing stresses induced by the fixed loads may be employed for the construction of the influence line.

Example

Construct the influence lines for the forces in the members which are marked in figure 2 – O_4, U_4, D_3, V_2, D_6 and D_7.

1. Influence lines for main truss members

1.1 Influence line for O_4.

In order to construct the influence line for the force in O_4 we shall pass section I-I across three bars of the corresponding panel (Fig. 2).
When the unit load is at the right hand side of joint 4, it is more convenient to consider the equilibrium of the left hand part of the truss as free body diagram (Fig. 2a). Free body diagram is the part of the truss into consideration with depicted external loads and introduced unknown member forces. This part is acted upon solely by the support reaction A. Placing the origin of moments at point 4 and equating to zero ΣM of all forces acting to the left of section I-I we obtain:

$$\Sigma M_4 = 0 \quad O_4 h + A \cdot 2\lambda = 0 \rightarrow O_4 = \frac{-2\lambda}{h} A .$$

Thus, when the unit load is applied to the right of joint 4, the axial force in bar O_4 equals the left hand reaction A multiplied by a constant factor $\left(-2\lambda / h\right)$. It should be pointed also that $2\lambda \cdot A$ is numerically equal to the bending moment M_4^0 acting over the cross section of the equivalent simply supported beam, placed bellow the origin of moments (Fig. 3b).
It is clear from the above that as long as the load remains to the right of joint 4 the influence line will be the same as for reaction A multiplied by $\left(-2\lambda / h\right)$ (See Fig. 3a).
When the load is located at the left of joint 2 the force O_2 can be obtained by the equilibrium of the right hand side of the truss.

$$\Sigma M_4 = 0 \quad O_4 h + B \cdot \lambda = 0 \rightarrow O_4 = \frac{-\lambda}{h} B .$$

The force into consideration equals in this case to the right hand reaction B multiplied by $\left(-\lambda / h\right)$. Once again $\lambda \cdot B$ is the equivalent of the single beam bending moment M_4^0 acting over a section corresponding to point 4 (Fig. 3b).
If all the operations have been carried out correctly both the lines corresponding to both considered parts of the truss will intersect under joint 4 (Fig. 3a).
Another way of obtaining the same influence line is based on the relationship between the member force O_4 and the simply beam bending moment M_4^0.

$$O_4 = -M_4^0 / h .$$

The above example leads to the following conclusion: The *influence lines for members belonging to the lower or upper chords could be obtained by using influence lines for bending moment in the equivalent simply supported beam.* In order to do that the origin of moments for the member into consideration must lay between both the supports. If that is fulfilled the influence lines for such members equals the bending moment influence line divided by the lever arm of the member force about the origin of moments ($\pm M^0 / r$, r is the lever arm, in case of parallel chords r=h).

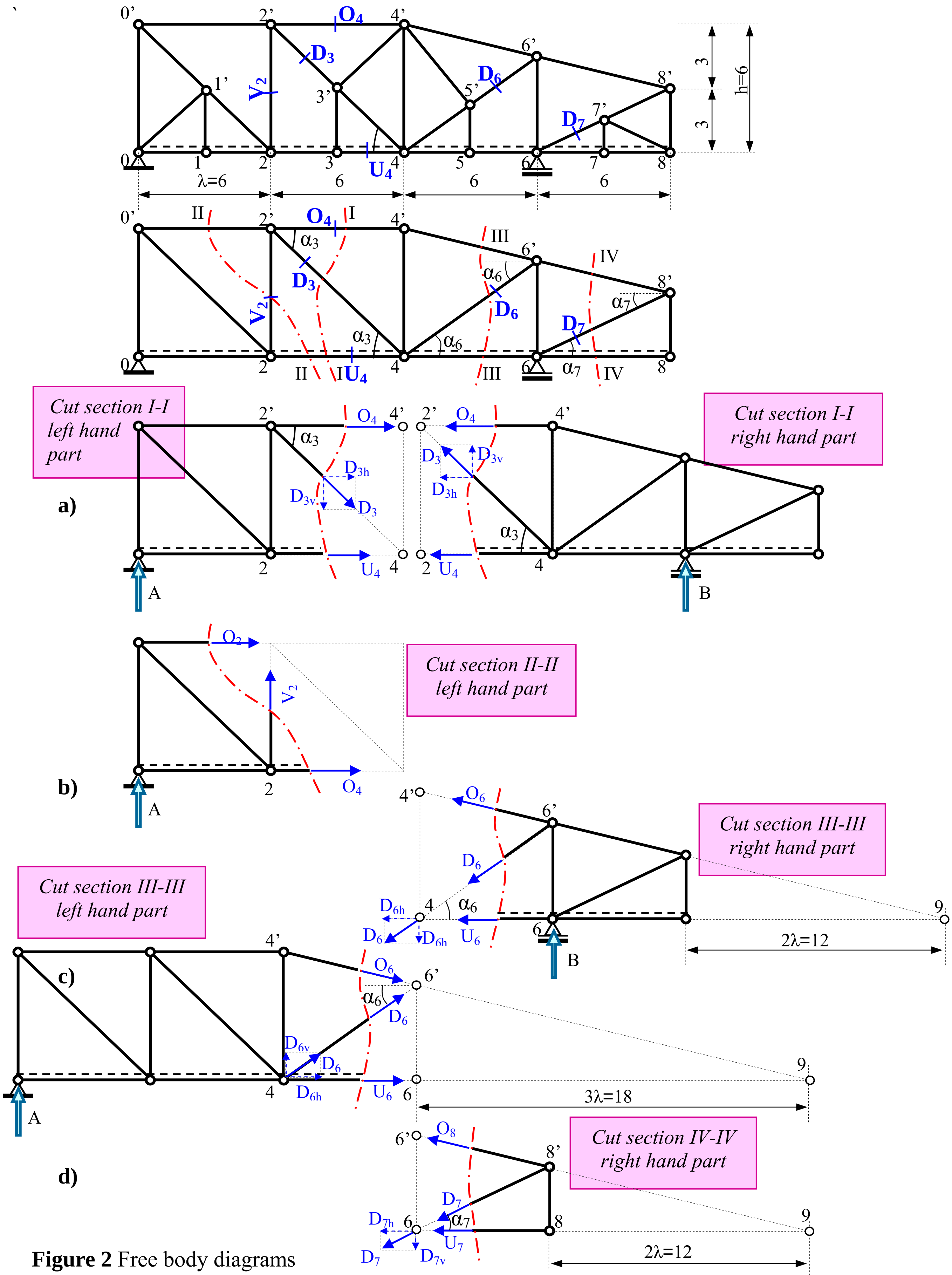


Figure 2 Free body diagrams

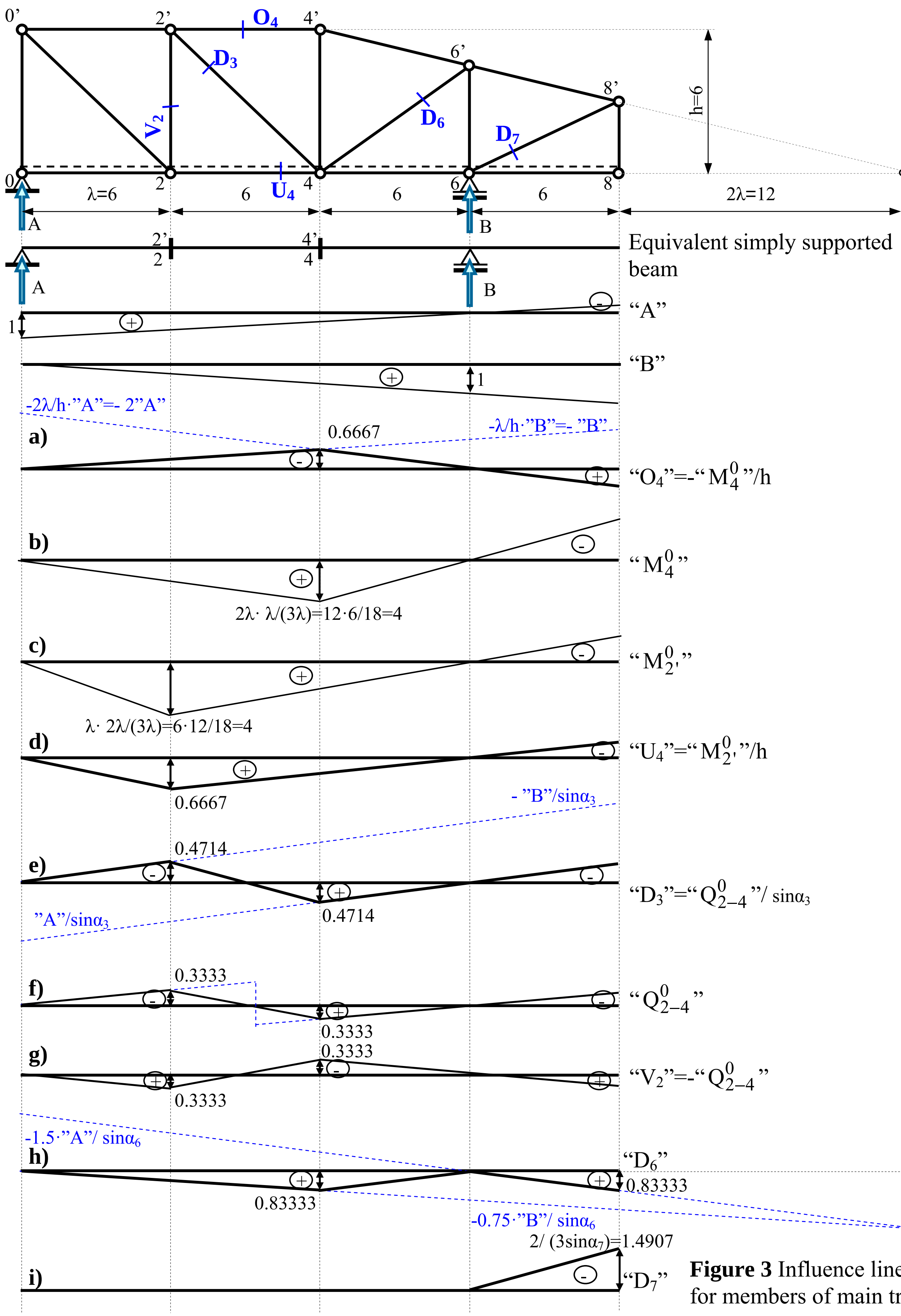


Figure 3 Influence lines for members of main truss

The sign of the influence lines ($\pm M^0 / r$) can be determined using analogy with the sign of bending moment in beam theory. When the force into consideration causes a moment about the point of origin in the same direction as positive bending moment in beam theory, then the relative influence line is $+M^0 / r$, and vice versa the influence line is $-M^0 / r$.

In line with the above conclusions let us construct the influence line for U_4.

1.2 Influence line for U_4.

The section cut is again I-I, the origin of moments is point 2' (Fig. 2a). Considering the left hand part of the truss the axial force U_4 causes a moment about 2' in counter-clockwise. The positive bending moment for left hand part of the equivalent simply supported beam is again counter-clockwise, therefore $U_4 = -M^0_{2'} / h$ (Fig. 3c,d).

1.3 Influence line for D_4.

The equilibrium of all the vertical projections of forces acting on the left hand portion of the truss, when the unit load F=1 travels between joints 4 and 8 (Fig. 2a), requires that:

$A - D_{3v} = 0$ or $D_{3v} = A$ respectively $D_3 = \dfrac{A}{\sin \alpha_3}$.

When the unit load is on the left hand side of joint 2 the same consideration relative to the right hand side portion of the truss reads:

$D_{3v} + B = 0$ or $D_{3v} = -B$ respectively $D_3 = \dfrac{-B}{\sin \alpha_3}$.

In such a way we have obtained the influence line D_3 in terms of both the reactions, respectively the portions of influence line between joints 0-2 and 4-8 (Fig. 3e). The portion between joints 2 and 4 is obtained by connecting these two points by a straight line.
If we consider again the analogy with simply supported beam for the left hand portion of a beam, for section belonging to panel 2-4, the shear force influence line equals to the left support reaction *A*. For the right hand portion, the shear force for the same section equals to *–B*. Consequently, the influence line for D_3 equals the shear force influence line in a simply supported beam (Fig. 3f) for a section belonging to the cut panel divided by sin*α* (*α* is the angle between force direction and horizontal axis). Bearing in mind that the loads are transferred to the truss panel nodes, the portion of Q between joints 2 and 4 should be replaced by a straight line.
Based on the above notes the following conclusion may be written: the influence lines for the diagonals between parallel chords can be constructed using the shear influence lines in the simply supported beam, for the cut panel, divided by sin*α*, where *α* is the angle between horizontal axis and the direction of member force. In general, the diagonals are $\pm Q / \sin \alpha$. The sign is $+Q / \sin \alpha$ when the vertical projection of the diagonal member force coincides with positive shear force for the part into consideration, otherwise the sign is $-Q / \sin \alpha$.
For vertical web member it is easy to find that the influence lines are equal to $\pm Q$ for the cut panel. When the direction of vertical member force, for the considered part, coincides with the positive shear force in the same part of a simply supported beam then $V = +Q$, and vice versa when the vertical force is opposite to the positive shear force $V = -Q$. It should be pointed out that this analogy is valid only when both the chords are parallel, and when the panel is between both the supports.

1.4 Influence line for V_2.

In accordance with the above statements let us construct the influence line for the member V_2. We shall pass a section II-II across three bars, and the section cut passes the road in panel 2-4.

The direction of member force V_2 for the left hand portion of the truss is opposite to the positive shear force. Thus, the force V_2 equals to $-Q_{2-4}$ (Fig. 3f,g).

1.5 Influence line for D_6.

What follows is the construction of influence line for D_6. In order to do that we shall pass a section cut III-III in panel 4-6 (Fig. 2). The origin of moments for D_6 is the intersection of both the chords, namely joint 9. When the moving loads is to the right of joint 6 we shall consider the equilibrium of the left hand portion of the truss:

$$A \cdot 6\lambda + D_{6v} \cdot 4\lambda = 0, \quad D_{6v} = D_6 \sin\alpha_6 \text{, wherefrom: } D_6 = -\frac{3A}{2\sin\alpha_5}.$$

When the load is to the left of joint 4 the force D_6 can be obtained by the equilibrium equation relative to the right hand portion of the truss:

$$B \cdot 3\lambda - D_{6v} \cdot 4\lambda = 0 \text{ or } D_6 = \frac{3B}{4\sin\alpha_5}.$$

Both the lines between points 0-4 and 6-8 of the influence line intersect each other below the origin of moments (below point 9) (Fig. 3h). The part of influence line between joints 4 and 6 will be obtained by connecting the values at these points.

1.6 Influence line for D_7.

The influence line for axial force in member D_7 will be derived by passing a cut IV-IV in panel 6-8 (Fig. 2). It is clear that when the unit load is on the left of joint 6, and we consider the equilibrium of the right hand portion of the truss, the member force D_5 is zero. This force will get nonzero value only when the unit load is at joint 8. Following the equilibrium of the right hand portion of the truss when the unit load is at joint 8, we get:

$$1 \cdot 2\lambda + D_{7v} \cdot 3\lambda = 0, \quad D_{7v} = D_7 \sin\alpha_7 \text{, wherefrom: } D_7 = -\frac{2}{3\sin\alpha_7}.$$

The influence line for member force D_7 is depicted in Fig. 3i.

2. Influence lines for auxiliary systems

The auxiliary systems remain idle as long as the unit load is outside of the panel which they reinforce. The axial forces of auxiliary members are nonzero only when the load is within the limits of that panel. This means that the influence lines for the members being a part of auxiliary systems are always triangular with nonzero value at the intermediate joint of the panel and zero value at the joints of the main truss.

The influence lines for members of auxiliary systems will be obtained by using the method of joints for the considered isolated auxiliary truss.

2.1 Influence lines for members D_3, U_4, O_4 and V_2.

All of these members belong to the auxiliary truss in panel 2-4 (Fig. 4). The members of this truss are zero when the unit load is in joint 2 and 4, and the members are loaded only when the unit load is in joint 3. Using the method of joints when the unit load is in joint 3 all force members will be derived. After that the influence lines for every member can be constructed (Fig. 4).

Likewise the influence lines for the other auxiliary members $\mathbf{D_6}$ and $\mathbf{D_7}$ are derived and depicted in Fig. 4.

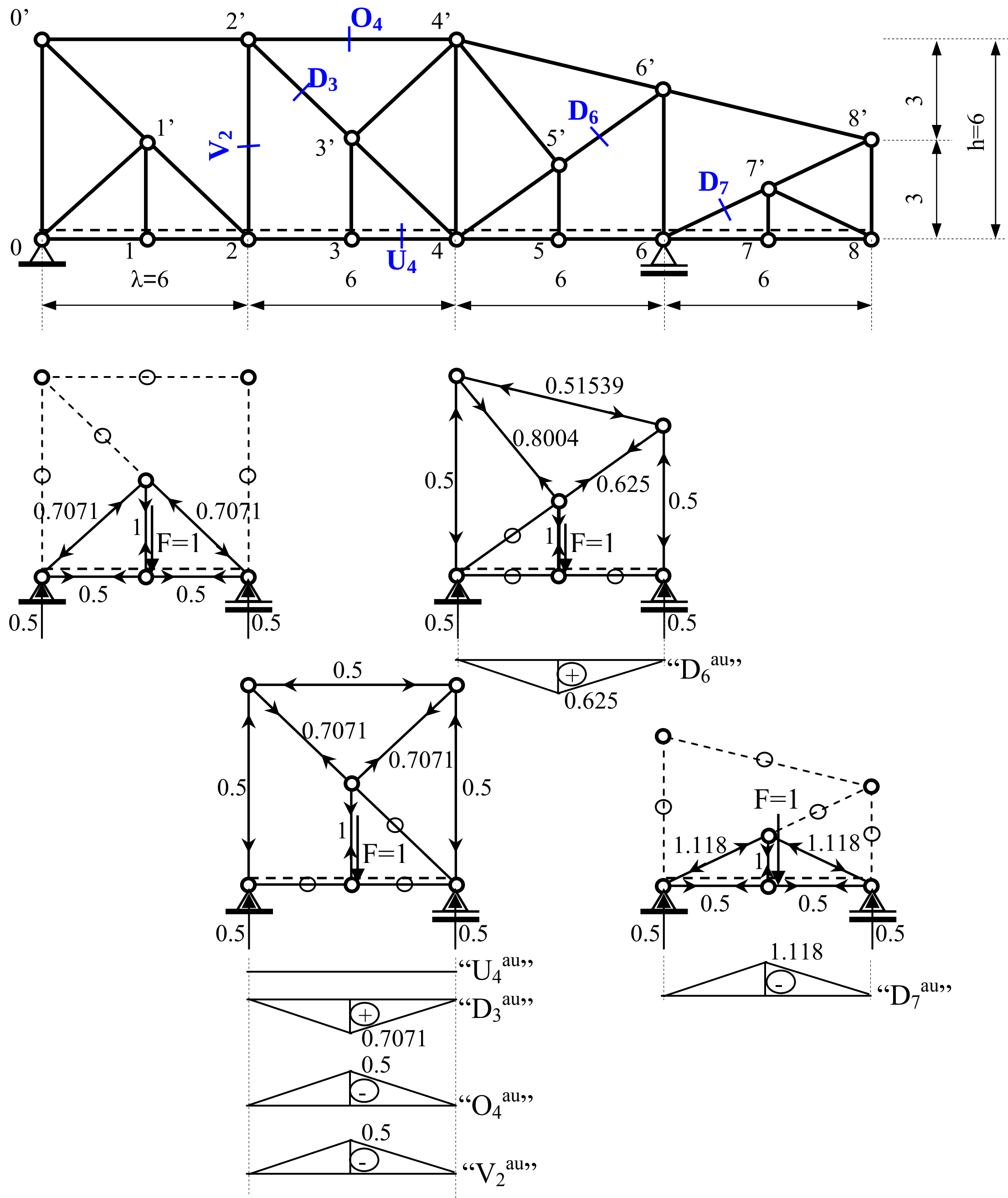


Figure 4 Influence lines for members belonging to auxiliary systems

3. Influence lines for the original truss

The influence lines for the members belonging simultaneously to the main truss and to the auxiliary systems should be obtained by the summation of influence lines pertaining to the main and auxiliary trusses (which have been already constructed separately). These influence lines are presented in Fig. 5.

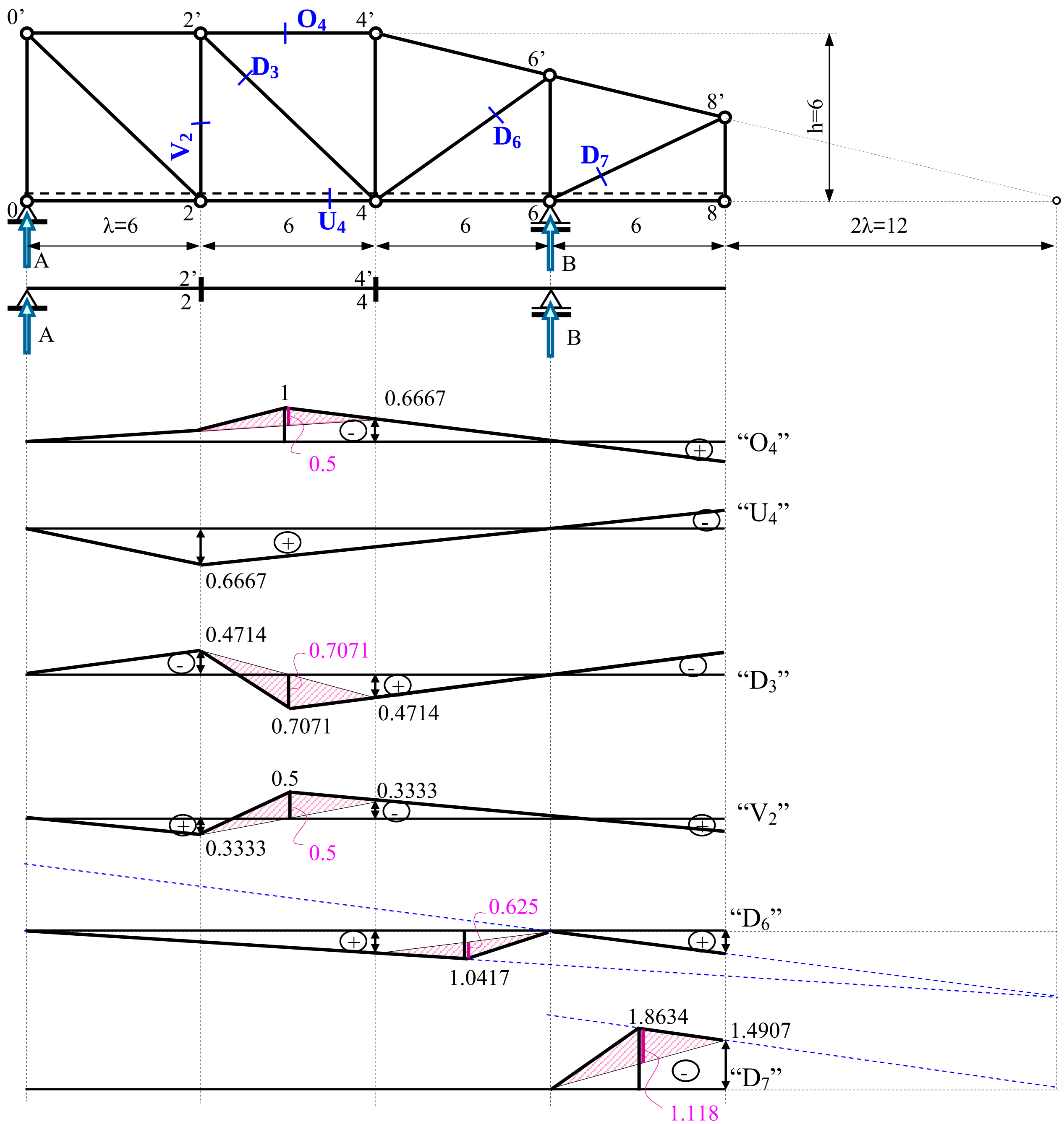


Figure 5 Influence lines for members as a part of the original truss

III. Kinematical method of influence lines construction

1. Basic concepts

The kinematic method of influence line construction for any given function is based on the most general principles of theoretical mechanics – the principle of virtual displacements.

In accordance with this principle, *the total work performed by any given system of forces along virtual displacements of a body in equilibrium must be zero*.

The virtual displacements are infinitely small. In that respect when a plate rotates an angle $d\varphi$ about point O, point A of a plate located on a distance r from point A shifts to A_2 along a circular arc (Fig. 6). Since the angle $d\varphi$ is very small, we may assume that point A moves along the tangent of the arc, neglecting distance A_1A_2 ($A_1A_2 \approx 0$). For the same reason $\mathrm{tg}\,d\varphi \approx d\varphi$ or $AA_1 = r \cdot d\varphi$.

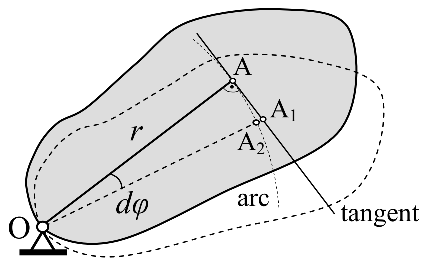


Figure 6 Virtual displacement of point A

Let us consider a system of three pin-connected plates simply supported to the ground (Fig. 7). The problem is to find the influence line for member force in plate 3 (plate 3 is unloaded; the internal forces are represented by axial force only). In accordance with the kinematic method we should remove the constraint representing the required force. In that respect the plate 3 is replaced by its axial force X. The system is instantaneously unstable (mechanism). Plate 1 is fixed at point (1), which is a main pole for this plate likewise plate 2 can be rotated about point (2). Plates 1 and 2 can rotate one another about their relative pole (1,2). The unit load is represented by force F. Let us assume that the virtual rotation of plate 1 is φ_1 the rotation of plate 2 is φ_2.

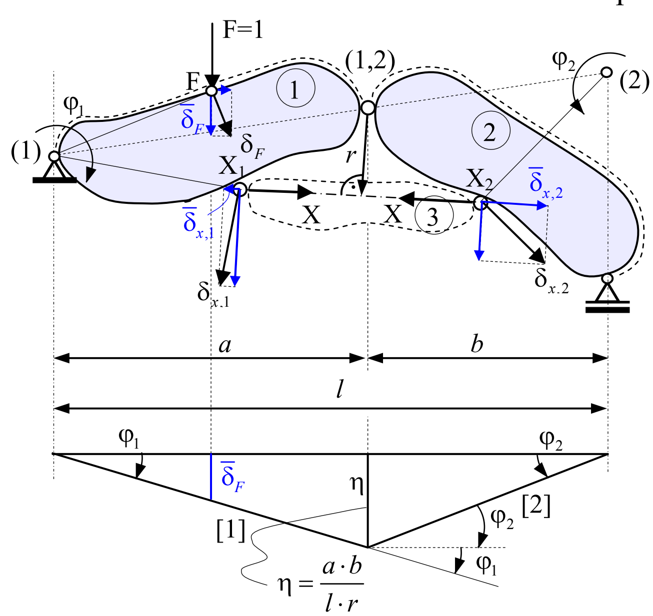


Figure 7 Vertical displacements graphics of instantaneously unstable system

The full virtual displacement of point *F*, where the force *F* is applied, is normal to line (1)*F* denoted δ_F. The full virtual displacement of points X_1 and X_2 are respectively $\delta_{x,1}$ and $\delta_{x,2}$ normal to lines connecting this points with main poles (1) and (2). Since the system is in equilibrium the total virtual work performed by system of forces must be zero:

$$F \cdot \overline{\delta}_F + X \cdot \overline{\delta}_{x,1} + X \cdot \overline{\delta}_{x,2} = 0\,,$$

where $\overline{\delta}_F$ is the projection of the full displacement δ_F in the direction of the applied unit load F, $\overline{\delta}_{x,1}$ and $\overline{\delta}_{x,2}$ are projections of the displacements $\delta_{x,1}$ and $\delta_{x,2}$ in the direction of member force X. $\overline{\delta}_x = \overline{\delta}_{x,1} + \overline{\delta}_{x,2}$ is relative displacement of both the member forces X, or:

$-F \cdot \overline{\delta}_F = X \cdot \overline{\delta}_x$, and since F=1 it follows that $X = -\dfrac{\overline{\delta}_F}{\overline{\delta}_x}$.

Therefore, if we introduce a unit relative displacement $\overline{\delta}_x$ in the direction of eliminated constraint, in such a way that the force performs a negative work ($\overline{\delta}_x = -1$), the member force X will be equal to the vertical displacement of the point where the unit load is applied $\overline{\delta}_F$ ($X = \overline{\delta}_F$). This relation postulates the Muller-Breslau principle: *The influence line for any constraint in a structure is the graph of vertical displacements of points of road lane obtained by application of unit virtual displacement at the point (points) of application of the constraint*.

The sequence in which the construction of influence line should be carried out is as follows:

1) Eliminate the constraint corresponding to the function under consideration and replace it by the relevant force (forces) in order to establish equilibrium;
2) Introduce a unit displacement in the mechanism in the direction of eliminated constraint in such a way that the force, replacing this constraint, performs negative work;
3) Draw the graph of virtual displacement for the mechanism obtained. This graph is the influence line in demand.

Main features of vertical displacements graph of the road lane are as follows:

1) Every plate of the road lane corresponds to a straight line in the vertical displacements graphics;
2) This line intersects the reference line (graph axis) under the main pole of the plate;
3) The lines from the vertical displacements graphics corresponding to the two plates from the road lane of the mechanism intersect one another under the relative pole (centre) of the plates.

2. Calculation of influence lines ordinates

The relative displacement $\overline{\delta}_x$ in the direction of eliminated constraint can be expressed as (Fig. 7):

$$\overline{\delta}_x = \varphi_{1,2} \cdot r\,,$$

where $\varphi_{1,2}$ is the relative rotation of plates 1 and 2; *r* is the lever arm of forces X about the relative pole (1,2).

$$\overline{\delta}_x = \overline{\delta}_{x,1} + \overline{\delta}_{x,2} = \left(\varphi_1 + \varphi_2\right) \cdot r\,.$$

According to vertical displacements graph (Fig. 7):

$\varphi_1 \cdot a = \varphi_2 \cdot b = \eta$ or $\varphi_2 = \dfrac{a}{b}\varphi_1$.

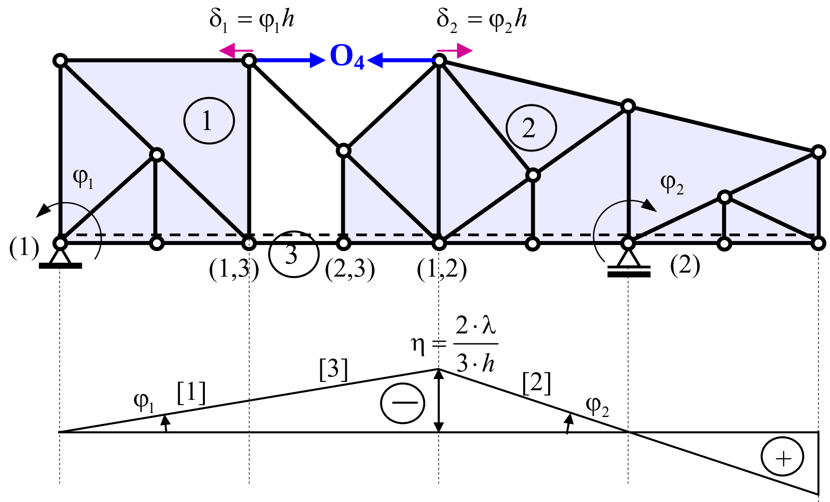
By replacing last expression in the above equation for $\bar{\delta}_x$ and bearing in mind that $\bar{\delta}_x = 1$ we get:

$\left(1+\dfrac{a}{b}\right)\varphi_1 \cdot r = 1$ respectively $\varphi_1 = \dfrac{b}{(a+b)r}$.

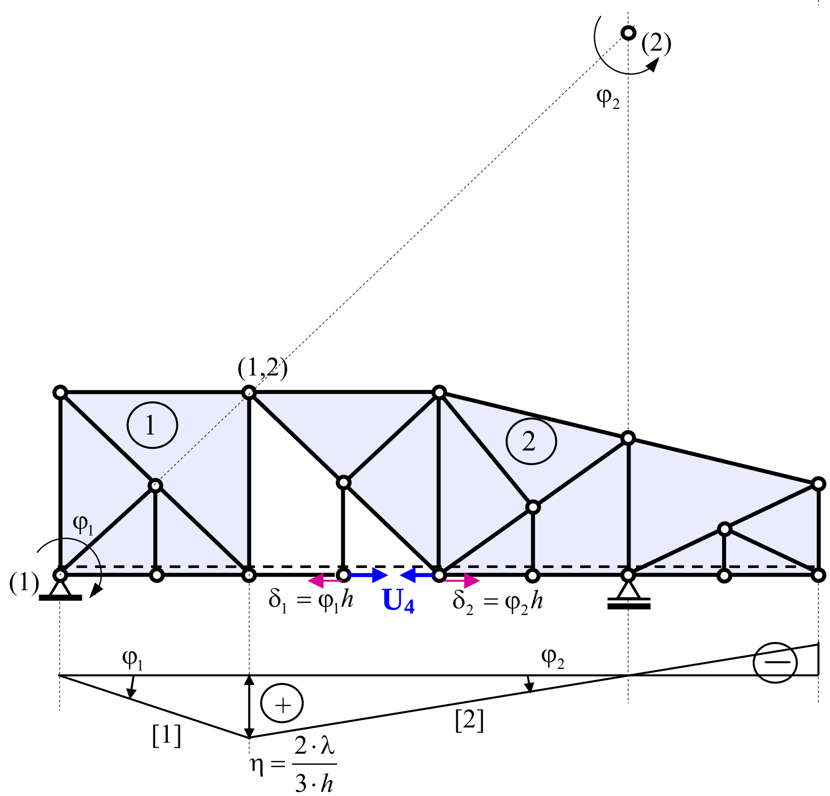
Finally:

$$\eta = \frac{a \cdot b}{l \cdot r}.$$

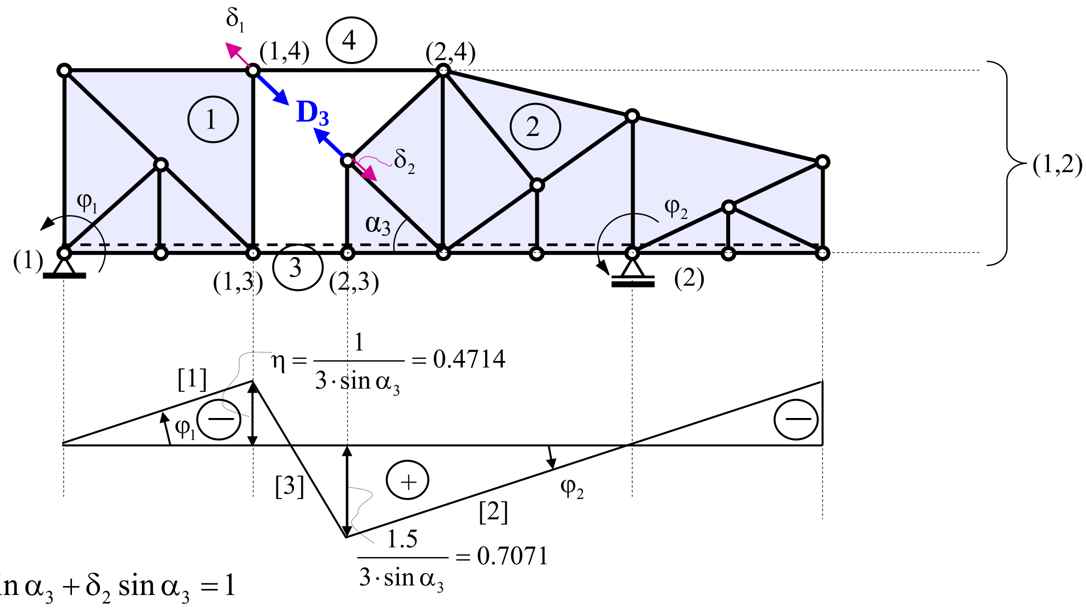
3. Kinematical construction of truss members influence lines.



$$\varphi_1 h + \varphi_2 h = 1$$
$$\eta = \varphi_1 \cdot 2\lambda = \varphi_2 \lambda$$
$$\varphi_2 = 2\varphi_1$$
$$\varphi_1 h + 2\varphi_1 \cdot h = 1$$
$$\varphi_1 = \frac{1}{3h}$$
$$\eta = \frac{2\lambda}{3h}$$



$$\varphi_1 h + \varphi_2 h = 1$$
$$\eta = \varphi_1 \cdot \lambda = \varphi_2 \cdot 2\lambda$$
$$\varphi_1 = 2\varphi_2$$
$$2\varphi_2 h + \varphi_2 \cdot h = 1$$
$$\varphi_2 = \frac{1}{3h}$$
$$\eta = \frac{2\lambda}{3h}$$



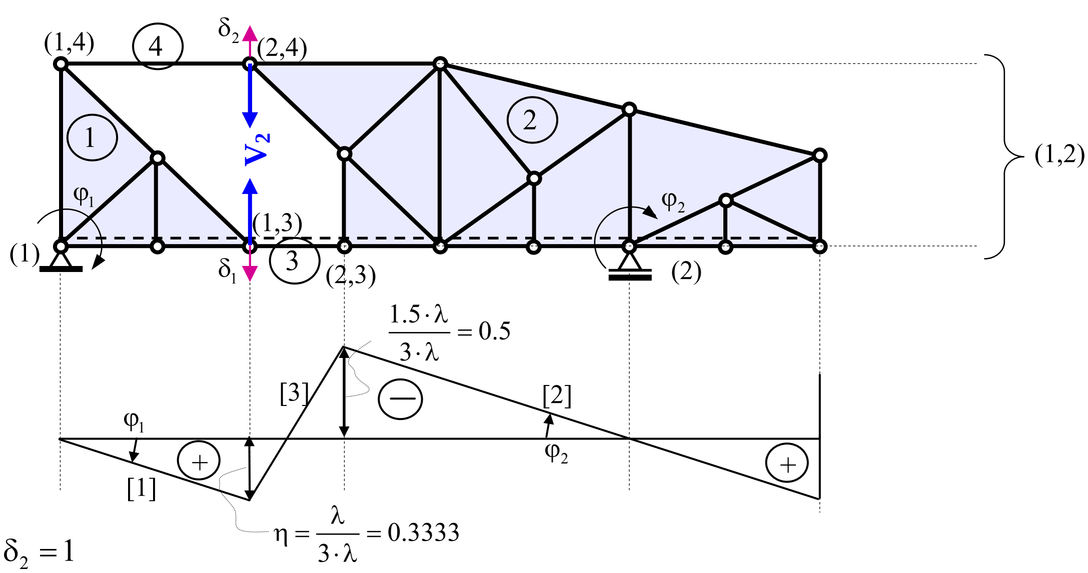
$$\delta_1 \sin\alpha_3 + \delta_2 \sin\alpha_3 = 1$$

$$\delta_1 + \delta_2 = \frac{1}{\sin\alpha_3}$$

$$\varphi_1 = \varphi_2 = \varphi$$

$$\lambda\cdot\varphi + 2\lambda\cdot\varphi = \frac{1}{\sin\alpha_3}$$

$$\varphi = \frac{1}{3\lambda\cdot\sin\alpha_3} \qquad \eta = \varphi\cdot\lambda = \frac{\lambda}{3\lambda\cdot\sin\alpha_3}$$

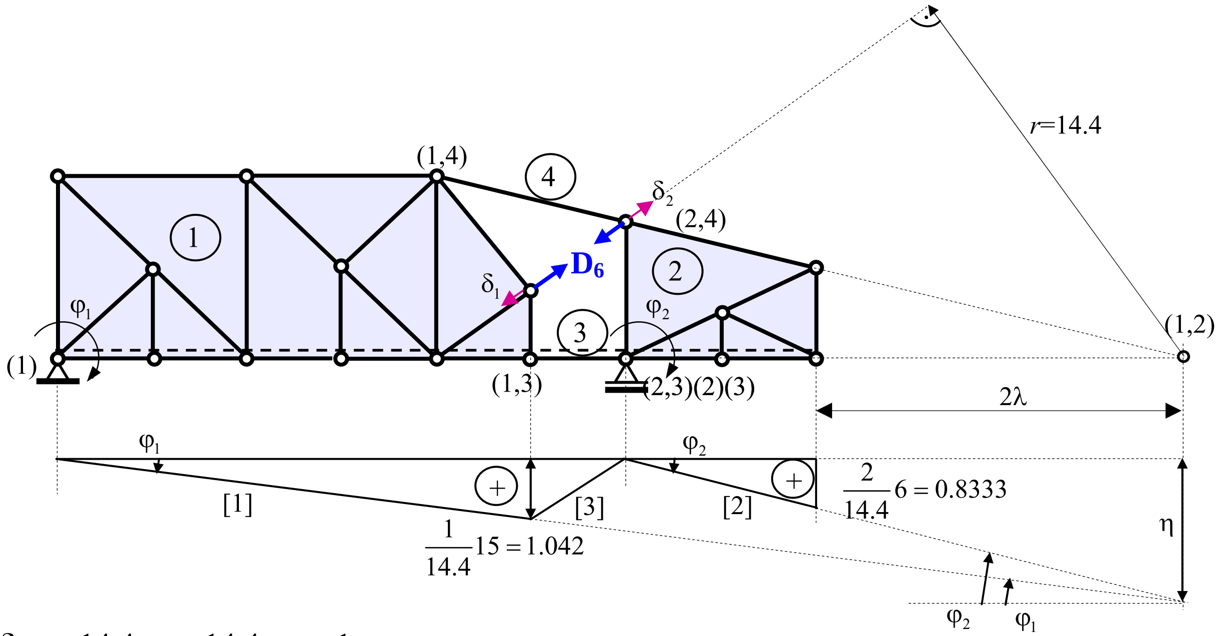


$$\delta_1 + \delta_2 = 1$$

$$\varphi_1 = \varphi_2 = \varphi$$

$$\lambda\cdot\varphi + 2\lambda\cdot\varphi = 1$$

$$\varphi = \frac{1}{3\lambda} \qquad \eta = \varphi\cdot\lambda = \frac{\lambda}{3\lambda}$$

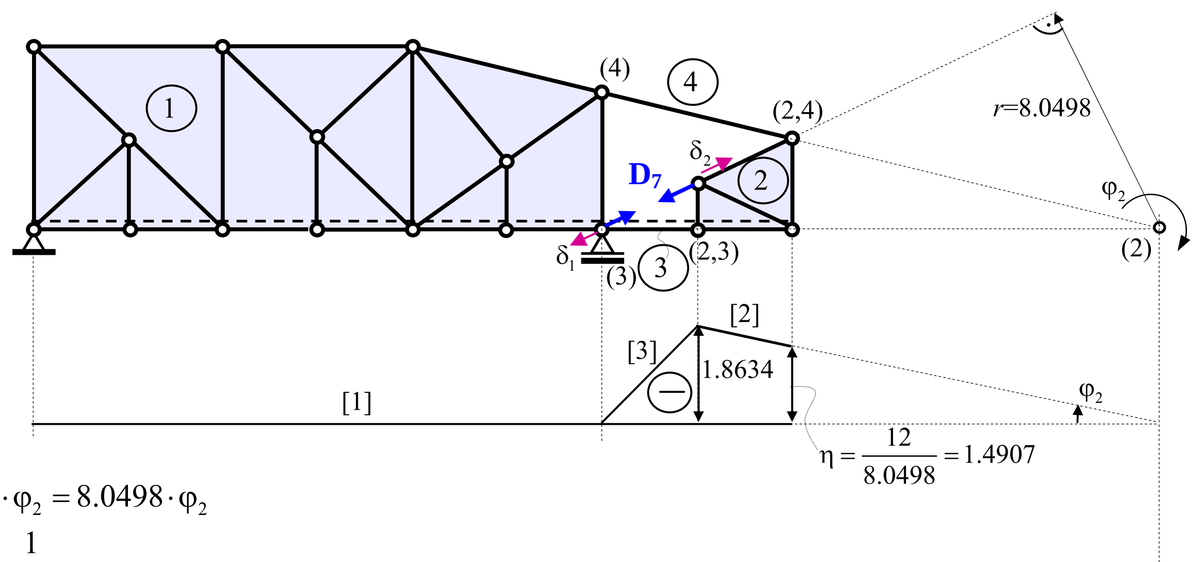


$$\delta_1 + \delta_2 = -14.4 \cdot \varphi_1 + 14.4 \cdot \varphi_2 = 1$$

$$\eta = 6\lambda \cdot \varphi_1 = 3\lambda \cdot \varphi_2$$

$$\varphi_2 = 2\varphi_1$$

$$-14.4 \cdot \varphi_1 + 14.4 \cdot 2\varphi_1 = 1 \qquad \varphi_1 = \frac{1}{14.4}, \quad \varphi_2 = \frac{2}{14.4}$$



$$\delta_2 = r \cdot \varphi_2 = 8.0498 \cdot \varphi_2$$

$$\varphi_2 = \frac{1}{8.0498}$$

$$\eta = 2\lambda \cdot \varphi_2 = \frac{12}{8.0498} = 1.4907$$

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