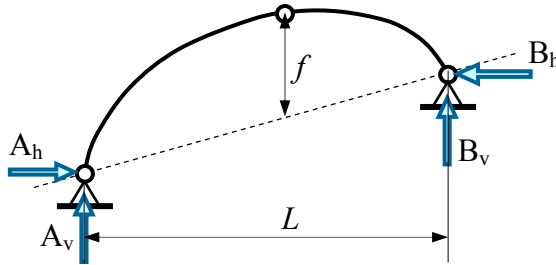


Analysis of three-hinged arches subjected to moving loads

A three-hinged system consists of two plates, connected together by means of a hinge with two hinged supports resting to the ground.

When the plates consist of curved bars the system is called three-hinged arch; in the case these bars are straight or L shaped, the system will be called a three-hinged bent or frame.

The distance L between the centres of the hinges at the supports is called the span of the arch. The distance f from the centre of the intermediate hinge to the straight line passing through the ground hinges is called its rise - f .



The reactions of three-hinged arch will be fully determined by four parameters, for instance, the amount of the reactions A_h , A_v , B_h and B_v . These may be obtained from the three equilibrium equations of external forces acting upon the system and from a fourth equation, expressing that the moment of all the external forces acting to the left or to the right of the intermediate hinge about its centre must be zero.

I. Influence lines for support reactions.

1. Vertical reactions

Let us assume that a three-hinged arch carries a unit load $F=1$ applied at a distance x from the left hand support, and let us write the equilibrium equation of the moments of all the forces about the support joints A and B (support pins) (Fig. 1).

$$\Sigma M_B = 0 \quad A_v \cdot l - 1 \cdot (l - x) = 0; \quad \Sigma M_A = 0 \quad B_v \cdot l - 1 \cdot x = 0$$

so, respectively:

$$A_v = \frac{l-x}{l}; \quad B_v = \frac{x}{l}.$$

The obtained expressions for A_v and B_v are absolutely the same as those for the reactions of a simply supported beam with length l . This means that the influence lines for A_v and B_v do not differ from the influence lines for the support reactions of the respective simple beam.

2. Horizontal reactions

The third equilibrium equation of the external forces is:

$$\Sigma H = 0 \quad \text{wherfrom } A_h - B_h = 0 \text{ or } A_h = B_h = H.$$

In addition the fourth equation can be used, expressing that the bending moment at the hinge C equals to zero, or in other words, the sum of moments of all the external forces acting to the right or to the left of this hinge about its centre is zero.

Let us assume that the unit force moves from point A to the hinge C, and let us consider the equilibrium of the right hand portion of the arch.

$$\Sigma M_C^{RP} = 0, \quad B_v \cdot l_2 - H \cdot f = 0,$$

$$H = \frac{B_v \cdot l_2}{f}$$

Now, let us consider the case when the unit load is located between points C and B, and let us write the equilibrium of moments of the left hand part about point C:

$$\Sigma M_C^{LP} = 0 \quad A_v \cdot l_1 - H \cdot f = 0,$$

$$H = \frac{A_v \cdot l_1}{f}$$

It should be pointed out that the expressions $B_v \cdot l_2$ and $A_v \cdot l_1$ coincide with the expressions of influence lines for the bending moment in equivalent simply supported beam for section c , below the intermediate hinge. In other words:

$$"H" = \frac{"M_c^0"}{f},$$

where " M_c^0 " is the influence line for bending moment of section c in a simply supported beam.

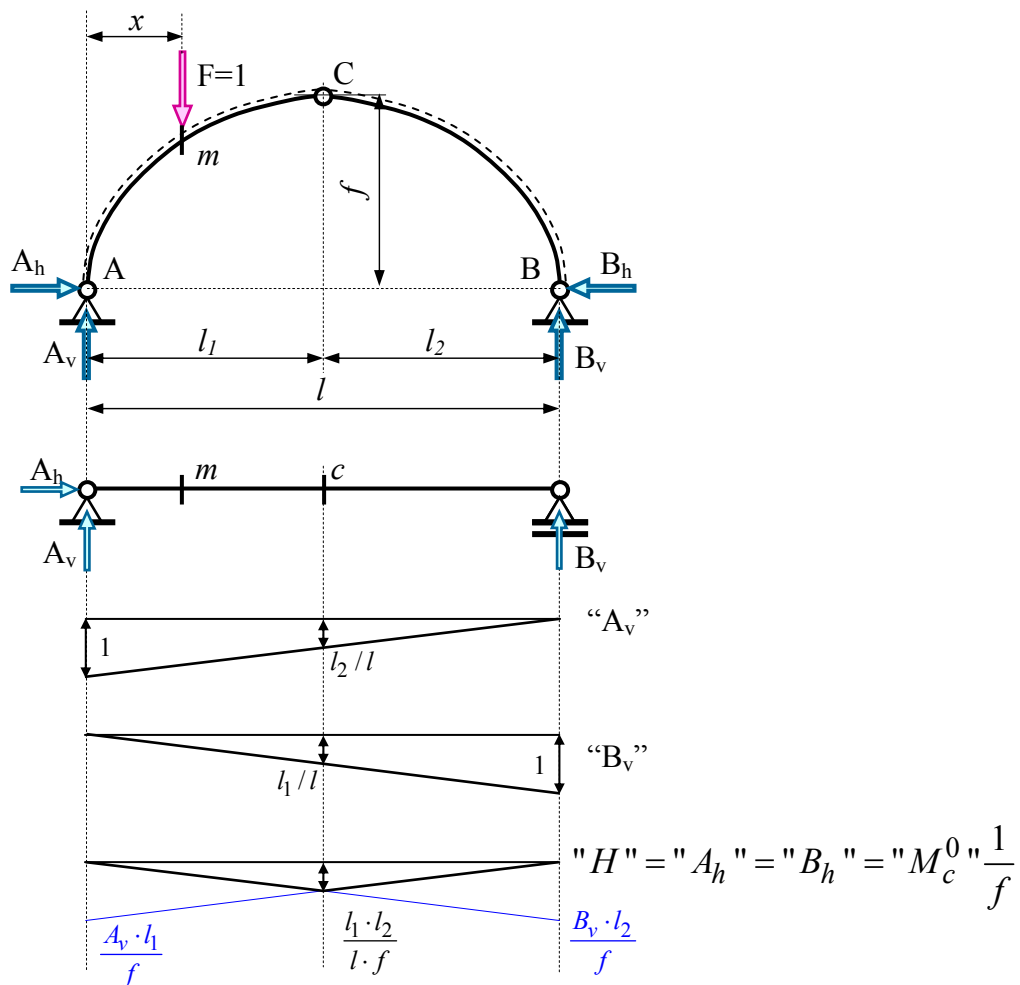


Figure 1 Influence lines for the support reactions

II. Influence lines for the internal forces

At first consider, the case when the unit load is between point A and section m . In this case we shall examine the equilibrium of the right hand portion of the arch (Fig. 2).

$$\Sigma M_m^{RP} = 0, \quad B_v \cdot (l - x_m) - H \cdot r - M = 0, \quad M = \underbrace{B_v \cdot (l - x_m)}_{M_m^0} - H \cdot r.$$

The expression $B_v \cdot (l - x_m)$ is the same as for the bending moment of section m in simply supported beam. Thus, the influence line for bending moment at point m is the influence line for the bending moment at the equivalent section in the simple beam minus the influence line of the horizontal reaction multiplied by r , or:

$$M_m = M_m^0 - H \cdot r.$$

The same expression will be derived if the unit load is on the right of section m , and we consider the equilibrium of the left hand part.

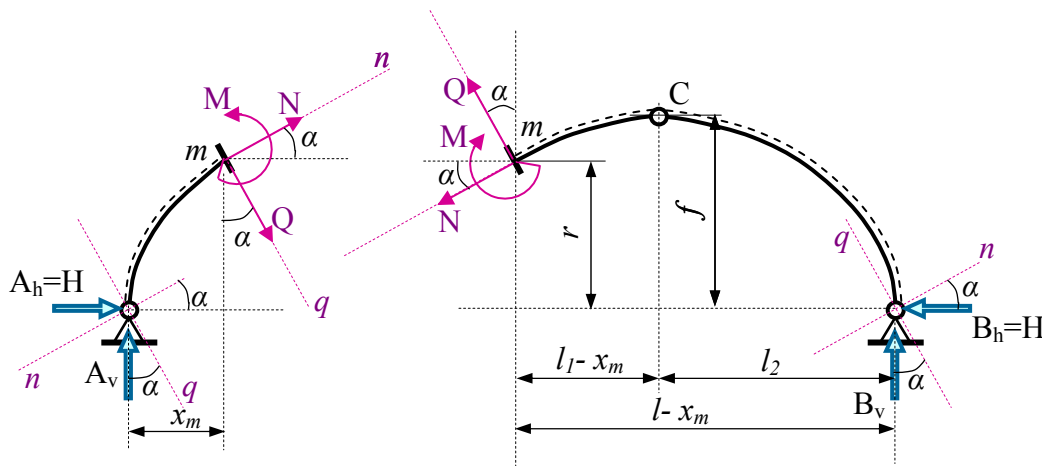


Figure 2 Internal forces at section m for the left and the right hand parts

The influence line for the axial force at section m is given by the equilibrium of all forces of the considered part in the axial force direction (Fig. 2):

$$N_m = B_v \cdot \sin \alpha - H \cdot \cos \alpha \quad \text{- unit load from A to } m;$$

$$N_m = -A_v \cdot \sin \alpha - H \cdot \cos \alpha \quad \text{- unit load from } m \text{ to B.}$$

The influence line for shear force at section m is obtained by equilibrium of all forces of the considered part in the shear direction (Fig. 2):

$$Q_m = -B_v \cdot \cos \alpha - H \cdot \sin \alpha \quad \text{when the unit load is from A to } m;$$

$$Q_m = A_v \cdot \cos \alpha - H \cdot \sin \alpha \quad \text{when the unit load is from } m \text{ to B.}$$

A complete drawing of the influence lines, for internal forces, is given in Fig. 3.

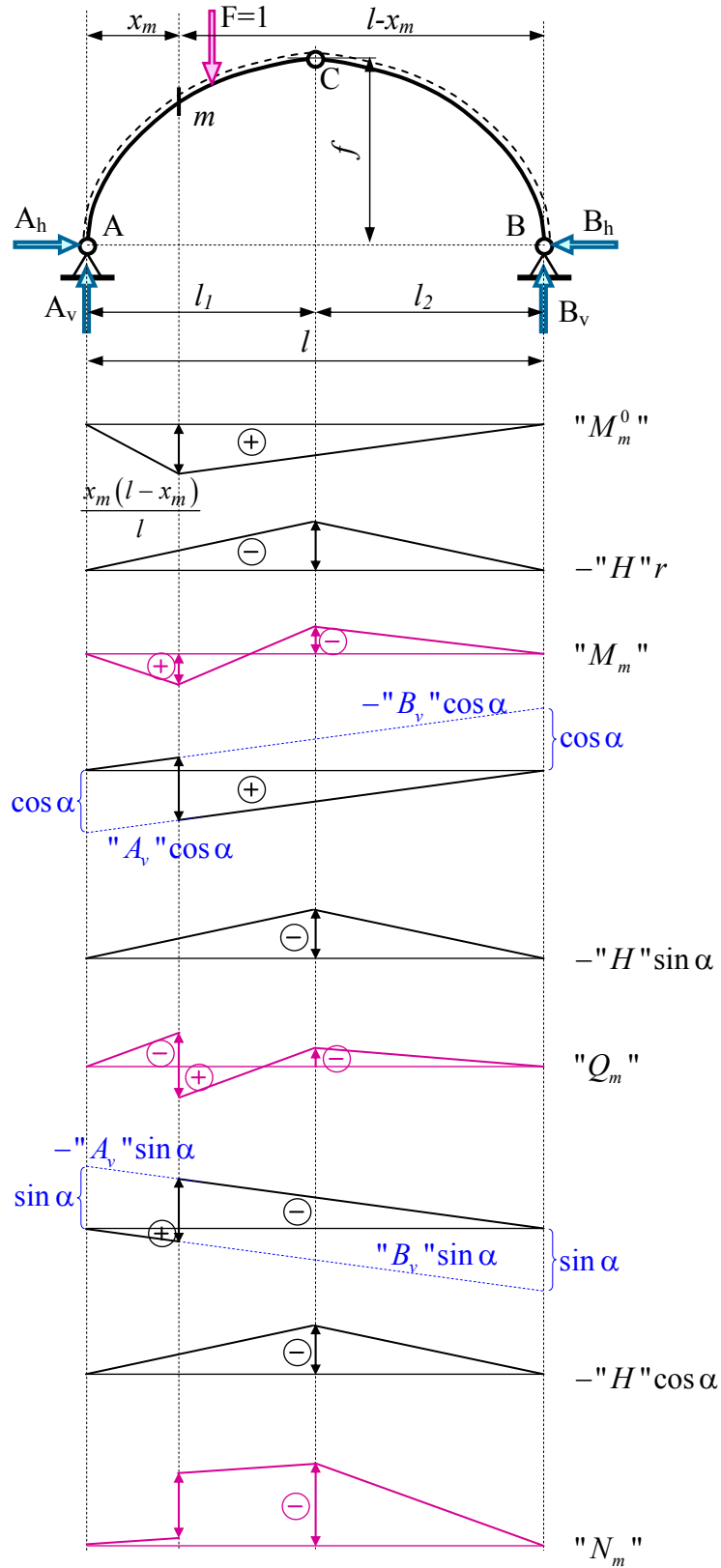


Figure 3 General view of the influence lines for internal forces

Influence lines for support reactions and internal forces in a real three-hinged arch.

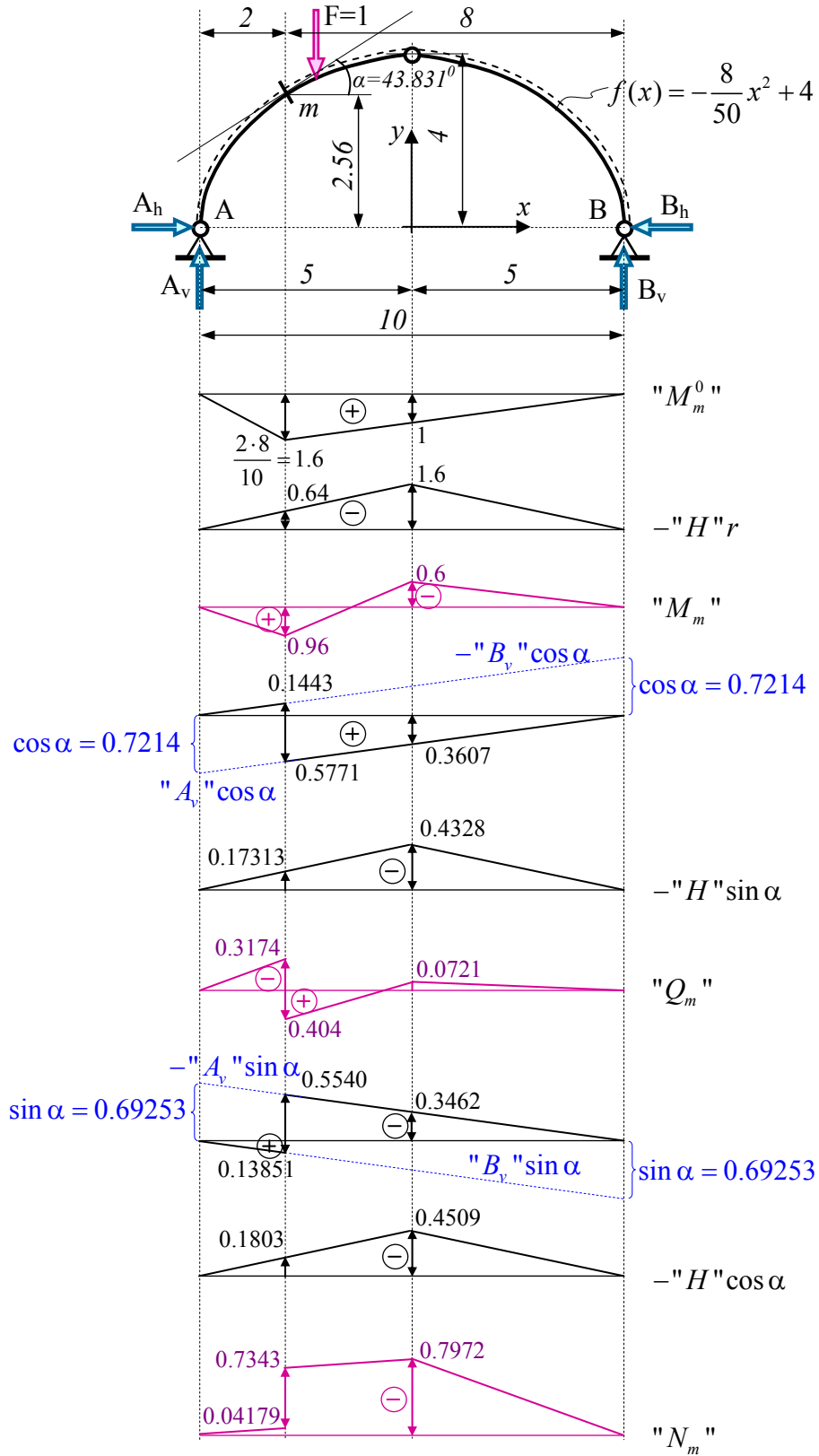


Figure 4 Influence lines for internal forces in a real example

Three hinged frame

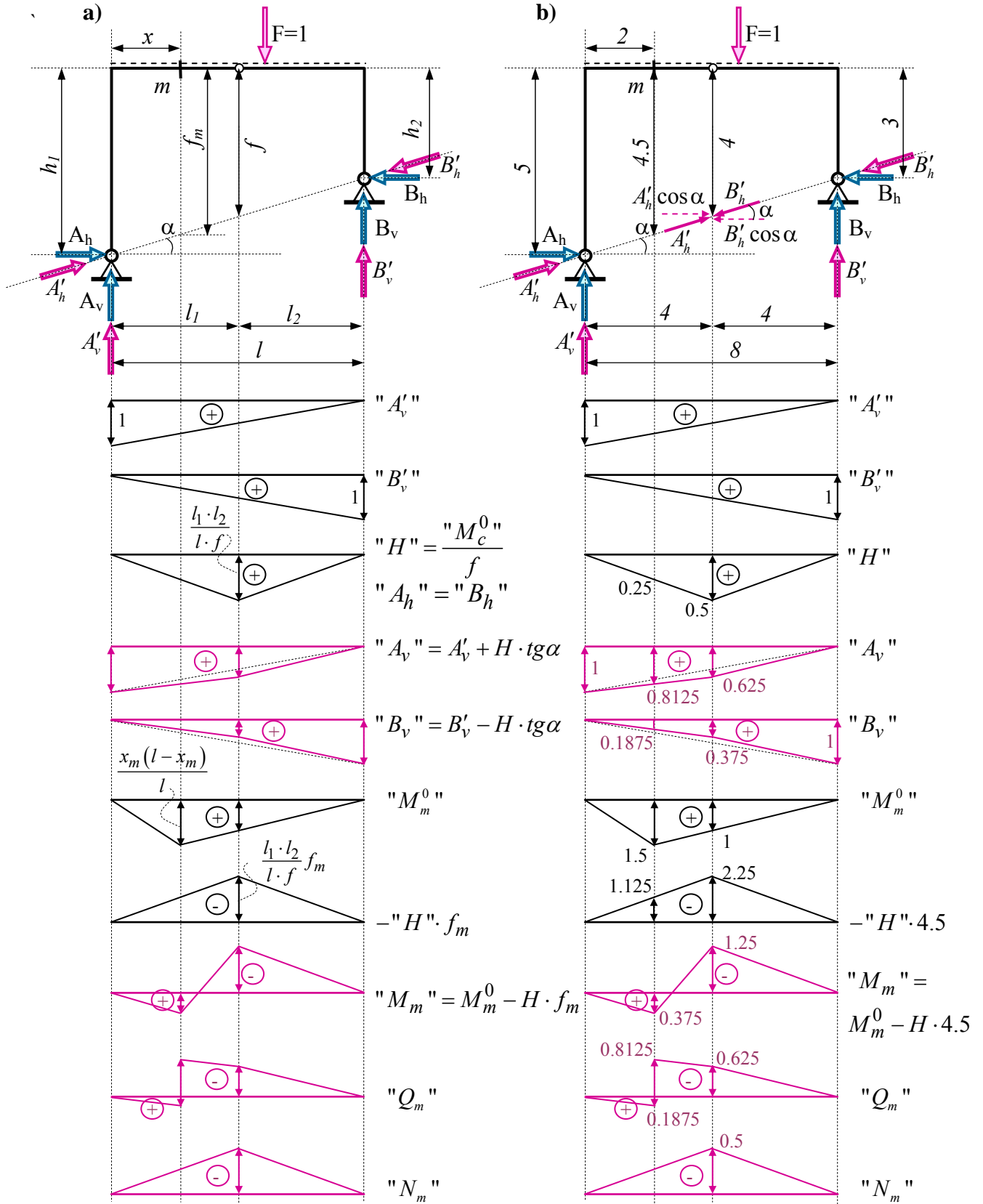


Figure 5 Influence lines for reactions and internal forces in a three hinged frame: **a)** general view; **b)** real example

I. Influence lines for support reactions.

In three-hinged frame shown in Fig. 5 the supports are at different levels. The unknown reactions are again A_v , A_h , B_v and B_h (Fig. 5a). If we use the equilibrium equation of the moments of all the forces about joint A we shall obtain one equation with two unknowns. The second equation with the same unknowns should be written for the equilibrium of the moments of the forces about intermediate hinge for the right hand part of the frame. In this case we shall generate two equations with two unknowns. The solution of system of two equations will make difficult the derivation of general expressions for the reaction influence lines.

This can be easily avoided if both reactions are resolved into components, one of which follow the line connecting the two supports A and B as shown in Fig. 5, the pink forces. The unknown reactions now are A'_v , A'_h , B'_v and B'_h , which can be derived by using the following equilibrium conditions:

$$\Sigma M_A = 0 \quad B'_v \cdot l - F \cdot x = 0 \quad B'_v = \frac{x}{l};$$

$$\Sigma M_B = 0 \quad A'_v \cdot l - F \cdot (l - x) = 0 \quad A'_v = \frac{l - x}{l};$$

$$\Sigma M_C^{RP} = 0 \quad B'_v \cdot l_2 - B'_h \cdot \cos \alpha \cdot f = 0 \quad \text{when } F=1 \text{ is from } A \text{ to } C;$$

$$\Sigma M_C^{LP} = 0 \quad A'_v \cdot l_1 - A'_h \cdot \cos \alpha \cdot f = 0 \quad \text{when } F=1 \text{ is from } C \text{ to } B;$$

$$\Sigma H = 0 \quad B'_h \cdot \cos \alpha = A'_h \cdot \cos \alpha = H.$$

The expressions for A'_v and B'_v are the same as those for the reactions of a simply supported beam with length l , and the corresponding graphics are given in Fig. 5.

The general expression for influence line for the horizontal reaction H , was obtained above, and can be derived as:

$$H = M_C^0 / f.$$

When all these components are determined, the initial horizontal and vertical components could be easily found using the following relationships:

$$A_v = A'_v + A'_h \cdot \sin \alpha = A'_v + H \cdot \operatorname{tg} \alpha;$$

$$B_v = B'_v - B'_h \cdot \sin \alpha = B'_v - H \cdot \operatorname{tg} \alpha;$$

$$A_h = B_h = H.$$

II. Influence lines for internal forces at section m .

At first the unit load moves between point A and section m . In this case we shall consider the equilibrium of the right hand part of the frame (Fig. 6):

$$\Sigma M_m^{RP} = 0 \quad M_m = B'_v \cdot (l - x_m) - H \cdot f_m;$$

$$\Sigma V = 0 \quad Q_m = -B_v;$$

$$\Sigma H = 0 \quad N_m = -B_h = -H.$$

Next, the unit load is between section m and point B . Following the equilibrium of the left hand part these expressions can be written (Fig. 6):

$$\Sigma M_m^{LP} = 0 \quad M_m = A'_v \cdot x_m - H \cdot f_m;$$

$$\Sigma V = 0 \quad Q_m = A_v;$$

$$\Sigma H = 0 \quad N_m = -A_h = -H.$$

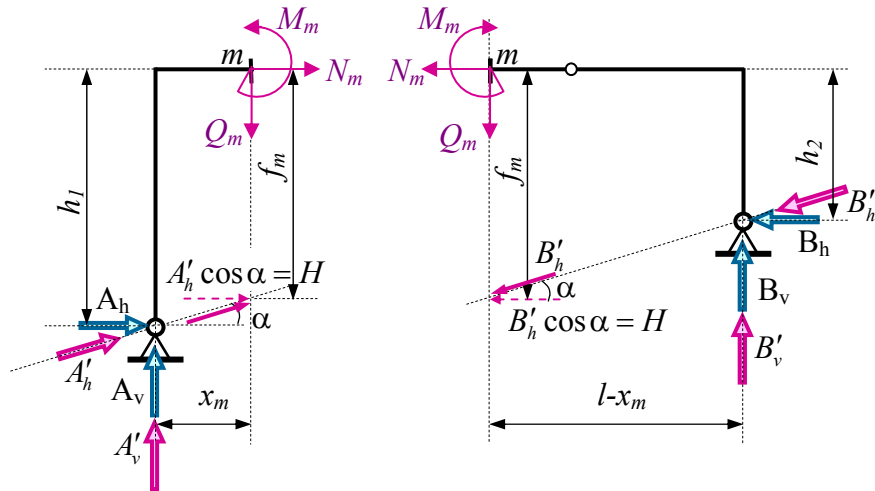


Figure 6 Internal forces at section m for the left and the right hand parts of a three-hinged frame

The expressions $B'_v \cdot (l - x_m)$ and $A'_v \cdot x_m$ are the same as those for the bending moment at section m in a simply supported beam. Therefore, the influence line for the bending moment in section m of the three hinged frame is:

$$M_m = M_m^0 - H \cdot f_m.$$

A general view of the influence lines, for internal forces at section m and support reactions, is given in Fig. 5a. The corresponding influence lines for the real three-hinged frame are depicted in Fig. 5b.

III. Determination of the internal forces by using influence lines

1. Case of concentrated loads.

The determination of any function, S , caused by a load F_i requires the measurement of the ordinate to the influence line for this function and its multiplication by the magnitude of the load. If the structure carries several loads at a time, the full value of the function in a section will be obtained by measuring the ordinates under each load, these ordinates being multiplied by the magnitude of the corresponding loads and the products summed up.

$$S = \sum_i F_i \cdot \eta_i.$$

2. Case of uniform loads.

A uniform load of intensity q is distributed along a certain length of a beam (Fig. 7). The problem is to calculate the value of function in a given section caused by the distributed load.

Let us replace the uniform load acting along an infinitely small length dx by a concentrated load $q \cdot dx$ (Fig. 7). The function into consideration due to this load ($q \cdot dx$) will be:

$$S = q \cdot dx \cdot \eta, \text{ where } \eta \text{ is the ordinate under the load } q \cdot dx.$$

Proceeding in the same way we can replace the whole load distributed along the beam by an infinitely great number of concentrated loads $q \cdot dx$. The function in the considered section will be then obtained by a summation of all the products $q \cdot dx \cdot \eta$, or:

$$S = \int_0^l q \cdot dx \cdot \eta = q \int_0^l \eta(x) dx .$$

$\int_0^l \eta(x) dx = \omega$ represents the area bounded by the influence line, the ordinates corresponding to the limits of loading and the x axis (the shaded area in Fig. 7).

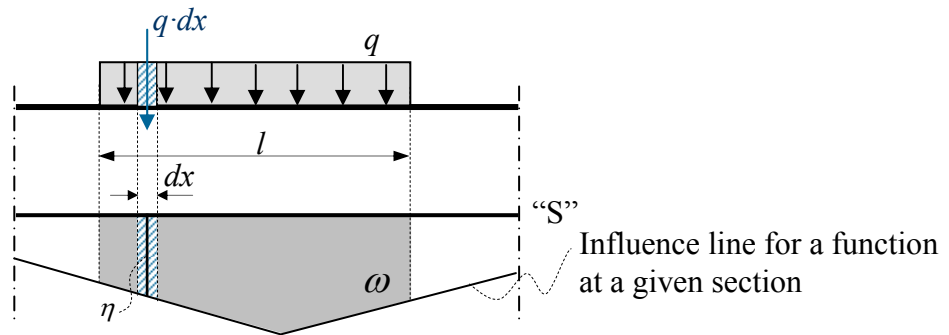


Figure 7 Case of uniform loads

Finally:

$$S = q \cdot \omega .$$

In order to determine the value of any function arising in a given section as a result of application of a uniform load, the intensity of this load must be multiplied by the area bounded by the influence line, the x axis and the ordinates passing through the load limits. When the influence line within the load limits changes sign the areas should be taken with their signs.

If the structure carries several distributed uniform loads at a time, the full value of the function in a section will be obtained by summation of the contribution of each load.

3. Case of concentrated moment.

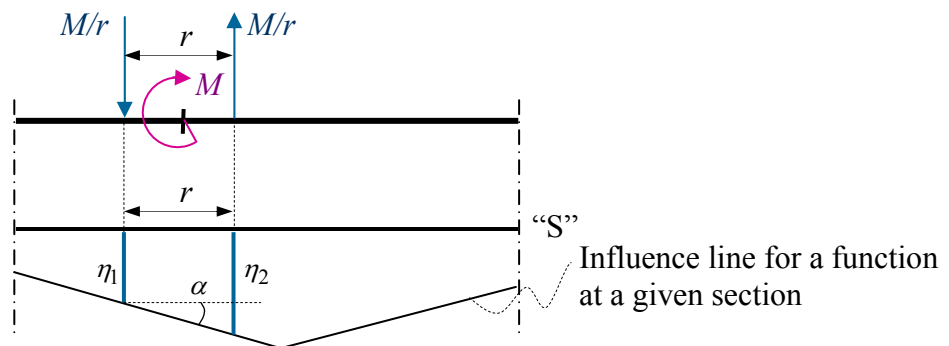


Figure 8 Case of concentrated moment

The concentrated moment, acting as an external load, can be replaced by a couple of forces with lever arm r (each force equals to M/r), as shown in Fig. 8. In this case the function becomes:

$$S = -F \cdot \eta_1 + F \cdot \eta_2 = F (\eta_2 - \eta_1) = \frac{M}{r} (\eta_2 - \eta_1) = M \cdot \operatorname{tg} \alpha .$$

When the moment and rotation angle are in the same direction the function is $S = +M \cdot tg\alpha$, when the moment and rotation angle are in the opposite directions the function is $S = -M \cdot tg\alpha$.

Finally, the value of any function arising in a given section as a result of application of some concentrated forces, a number of different uniform loads and some moments, could be obtained by the equation:

$$S = \sum_i F_i \cdot \eta_i + \sum_i q_i \cdot \omega_i + \sum_i M_i \cdot tg\alpha_i$$

4. Numerical example

Let us calculate the support reactions and internal forces at section m , for the three-hinged frame considered above, by using influence lines. The loads are depicted in Fig. 9.

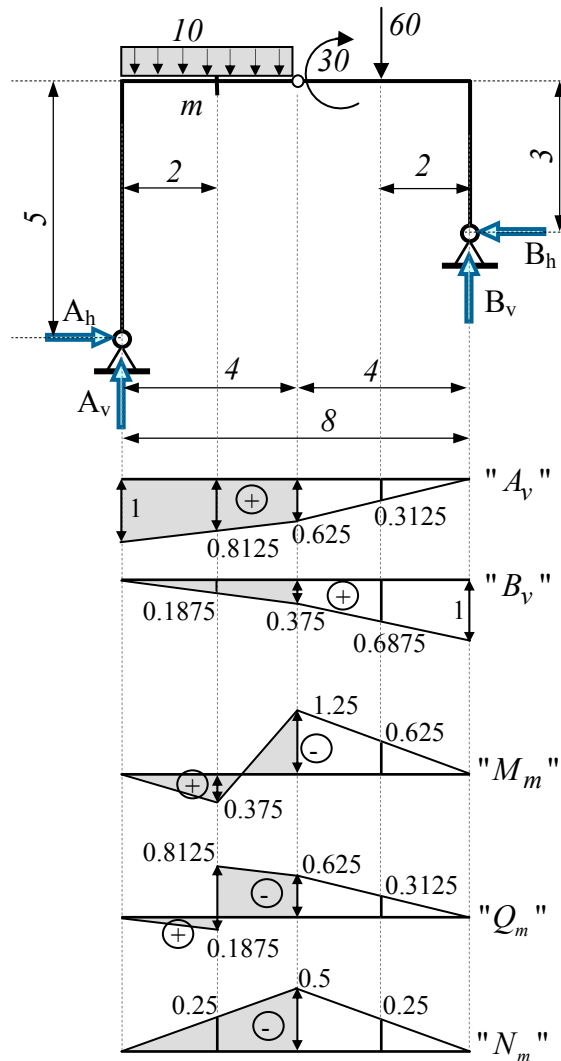


Figure 9 Three-hinged frame: loads and influence lines

In accordance with the obtained general expression for the value of any function, as a result of the application of different type of loads, the required values are as follows:

$$A_v = \frac{1+0.625}{2} \cdot 4 \cdot 10 - 30 \frac{0.625}{4} + 60 \cdot 0.3125 = 46.5625 \text{ kN} ;$$

$$B_v = \frac{0.375}{2} \cdot 4 \cdot 10 + 30 \frac{1-0.375}{4} + 60 \cdot 0.6875 = 53.4375 \text{ kN} ;$$

$$M_m = \frac{0.375}{2} \cdot 2 \cdot 10 + \frac{0.375-1.25}{2} \cdot 2 \cdot 10 - 30 \frac{1.25}{4} - 60 \cdot 0.625 = -33.125 \text{ kNm} ;$$

$$Q_m = \frac{0.1875}{2} \cdot 2 \cdot 10 - \frac{0.8125+0.625}{2} \cdot 2 \cdot 10 + 30 \frac{0.625}{4} - 60 \cdot 0.3125 = -26.5625 \text{ kN} ;$$

$$N_m = -\frac{0.5}{2} \cdot 4 \cdot 10 + 30 \frac{0.5}{4} - 60 \cdot 0.25 = -21.25 \text{ kN} .$$

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