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Beams on elastic foundation

I. Basic concepts.

The beam lies on elastic foundation when under the applied external loads, the reaction forces of the foundation are proportional at every point to the deflection of the beam at this point. This assumption was introduced first by Winkler in 1867.

Consider a straight beam supported along its entire length by an elastic medium and subjected to vertical forces acting in the plane of symmetry of the cross section (Fig. 1)

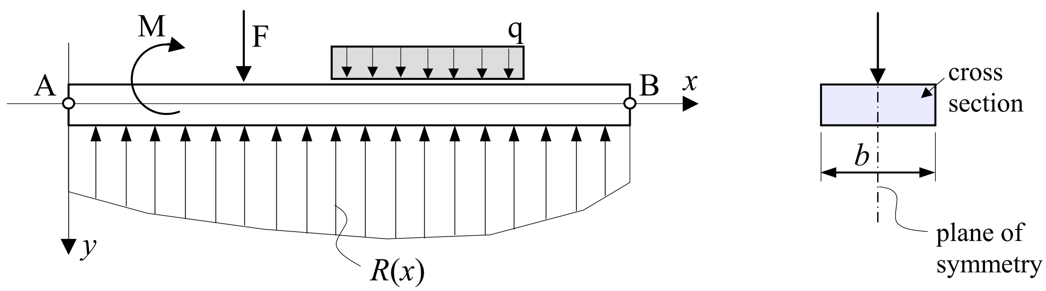

Figure 1 Beam on elastic foundation

Because of the external loadings the beam will deflect producing continuously distributed reaction forces in the supporting medium. The intensity of these reaction forces at any point is proportional to the deflection of the beam $y(x)$ at this point via the constant k:
$R(x)=k\cdot y(x)$.
The reactions act vertically and opposing the deflection of the beam. Hence, where the deflection is acting downward there will be a compression in the supporting medium. Where the deflection happens to be upward in the supporting medium tension will be produced which is not possible. In spite of all it is assumed that the supporting medium is elastic and is able to take up such tensile forces.
In other words the foundation is made of material which follows Hooke's law. Its elasticity is characterized by the force, which distributed over a unit area, will cause a unit deflection. This force is a constant of the supporting medium called ***the modulus of the foundation*** k_0 [kN/m^2/m].
Assume that the beam under consideration has a constant cross section with constant width b which is supported by the foundation. A unit deflection of this beam will cause reaction equal to $k_0\cdot b$ in the foundation, therefore the intensity of distributed reaction (per unit length of the beam) will be:
$R(x)= b\cdot k_0\cdot y(x)= k\cdot y(x)$,
where $k= k_0\cdot b$ is the ***constant of the foundation***, known as ***Winkler's constant***, which includes the effect of the width of the beam, and has dimension kN/m/m.

II. Differential equation of equilibrium of a beam on elastic foundation

Consider an infinitely small element enclosed between two vertical cross sections at distance dx apart on the beam into consideration (Fig. 2). Assume that this element was taken from a portion where the beam was acted upon by a distributed loading $q(x)$. The internal forces that arise in section cuts are depicted in Fig. 2.

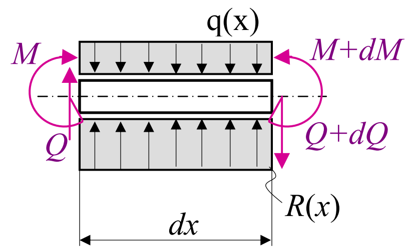

Figure 2 Differential element of length *dx*

Considering the equilibrium of the differential element, the sum of the vertical forces gives:

$$\Sigma V = 0 \quad Q - (Q + dQ) + \underbrace{R(x)}_{k \cdot y(x)} \cdot dx - q(x) \cdot dx = 0;$$

$$\frac{dQ}{dx} = k \cdot y - q \,.$$

Considering the equilibrium of moments along the left section of the element we get:

$$\Sigma M = 0 \quad dM - (Q + \underset{\to 0}{dQ}) \cdot dx - \underset{\to 0}{q \frac{dx^2}{2}} + \underset{\to 0}{r \frac{dx^2}{2}} = 0;$$

$$\frac{dM}{dx} = q \,.$$

Using now the well known differential equation of a beam in bending:

$$\frac{d^2 y}{dx^2} = -\frac{M}{EI},$$

it can be written:

$$\frac{dQ}{dx} = \frac{d^2 M}{dx^2} = -EI \frac{d^4 y}{dx^4} \,.$$

Finally, from the summation of the vertical forces $\Sigma V = 0$:

$$-EI \frac{d^4 y}{dx^4} = k \cdot y - q;$$

$$y^{IV} + 4 \underbrace{\frac{k}{4EI}}_{\alpha^4} \cdot y = \frac{q(x)}{EI} \,.$$

In the above equation the parameter α includes the flexural rigidity of the beam as well as the elasticity of the foundation. This factor is called the ***characteristic of the system*** with dimension length^{-1}. In that respect $1/\alpha$ is referred to as the so called ***characteristic length***. Therefore, $\alpha \cdot x$ will be an absolute number.

The differential equation of equilibrium of an infinitely small element becomes:

$$y^{IV} + 4\alpha^4 \cdot y = \frac{q(x)}{EI}, \qquad \alpha = \sqrt[4]{\frac{k}{4EI}}.$$

The solution of this differential equation could be expressed as:

$$y(x) = y_0(x) + v(x),$$

where $y_0(x)$ is the solution of homogeneous differential equation $y^{IV} + 4\alpha^4 \cdot y = 0$, $v(x)$ is a particular integral corresponding to q(x).

1. Solution of homogeneous differential equation

The characteristic equation of the differential equation under consideration is:

$$r^4 + 4\alpha^4 = 0.$$

After simple transformations this characteristic equation could be presented as:

$$r^4 + \left(2\alpha^2\right)^2 = r^4 - i^2 \cdot \left(2\alpha^2\right)^2 = \left[r^2 - i \cdot \left(2\alpha^2\right)\right]\left[r^2 + i \cdot \left(2\alpha^2\right)\right] = 0,$$

$$\left[r^2 + \alpha^2 - i \cdot \left(2\alpha^2\right) - \alpha^2\right]\left[r^2 + \alpha^2 + i \cdot \left(2\alpha^2\right) - \alpha^2\right] = 0$$

$$\left[r^2 + \alpha^2 - i \cdot \left(2\alpha^2\right) + i^2\alpha^2\right]\left[r^2 + \alpha^2 + i \cdot \left(2\alpha^2\right) + i^2\alpha^2\right] = 0$$

$$\left[r^2 + \left(\alpha - i \cdot \alpha\right)^2\right]\left[r^2 + \left(\alpha + i \cdot \alpha\right)^2\right] = \left[r^2 - i^2\left(\alpha - i \cdot \alpha\right)^2\right]\left[r^2 - i^2\left(\alpha + i \cdot \alpha\right)^2\right] = 0$$

$$\left[r - i\left(\alpha - i \cdot \alpha\right)\right]\left[r + i\left(\alpha - i \cdot \alpha\right)\right]\left[r - i\left(\alpha + i \cdot \alpha\right)\right]\left[r + i\left(\alpha + i \cdot \alpha\right)\right] = 0,$$

wherefrom the roots of the above characteristic equation are:

$$r_1 = \left(\alpha + i \cdot \alpha\right), \quad r_2 = \left(\alpha - i \cdot \alpha\right), \quad r_3 = \left(-\alpha + i \cdot \alpha\right), \quad r_4 = \left(-\alpha - i \cdot \alpha\right).$$

The general solution of homogeneous differential equation takes the form:

$$y_0(x) = A_1 \cdot e^{r_1 \cdot x} + A_2 \cdot e^{r_2 \cdot x} + A_3 \cdot e^{r_3 \cdot x} + A_4 \cdot e^{r_4 \cdot x};$$

$$y_0(x) = A_1 \cdot e^{(\alpha + i\cdot\alpha)\cdot x} + A_2 \cdot e^{(\alpha - i\cdot\alpha)\cdot x} + A_3 \cdot e^{(-\alpha + i\cdot\alpha)\cdot x} + A_4 \cdot e^{(-\alpha - i\cdot\alpha)\cdot x};$$

$$y_0(x) = A_1 \cdot e^{\alpha x} e^{i\alpha x} + A_2 \cdot e^{\alpha x} e^{-i\alpha x} + A_3 \cdot e^{-\alpha x} e^{i\alpha x} + A_4 \cdot e^{-\alpha x} e^{-i\alpha x}.$$

Using the well known Euler's expressions:

$$e^{i\alpha x} = \cos\alpha x + i \cdot \sin\alpha x,$$

$$e^{-i\alpha x} = \cos\alpha x - i \cdot \sin\alpha x.$$

the solution takes the form:

$$y_0(x) = A_1 \cdot e^{\alpha x}\left(\cos\alpha x + i \cdot \sin\alpha x\right) + A_2 \cdot e^{\alpha x}\left(\cos\alpha x - i \cdot \sin\alpha x\right) + A_3 \cdot e^{-\alpha x}\left(\cos\alpha x + i \cdot \sin\alpha x\right) +$$
$$+ A_4 \cdot e^{-\alpha x}\left(\cos\alpha x - i \cdot \sin\alpha x\right)$$

After simple regrouping of the members the expression becomes:

$$y_0(x) = e^{-\alpha x}\left[\underbrace{\left(A_3 + A_4\right)}_{B_1}\cos\alpha x + \underbrace{\left(iA_3 - iA_4\right)}_{B_2}\sin\alpha x\right] + e^{\alpha x}\left[\underbrace{\left(A_1 + A_2\right)}_{B_3}\cos\alpha x + \underbrace{\left(iA_1 - iA_2\right)}_{B_4}\sin\alpha x\right].$$

By introducing the new constants B_1-B_4 where $B_1 = (A_3 + A_4)$, $B_2 = (iA_3 - iA_4)$, $B_3 = (A_1 + A_2)$ and $B_4 = (iA_1 - iA_2)$ the solution can be written in a more convenient form:

$$y_0(x) = e^{-\alpha x}\left[B_1 \cos\alpha x + B_2 \sin\alpha x\right] + e^{\alpha x}\left[B_3 \cos\alpha x + B_4 \sin\alpha x\right]$$

Taking into account the expressions:

$\operatorname{ch}\alpha x = \frac{1}{2}\left(e^{\alpha x} + e^{-\alpha x}\right)$ and $\operatorname{sh}\alpha x = \frac{1}{2}\left(e^{\alpha x} - e^{-\alpha x}\right)$, respectively:

$e^{\alpha x} = \operatorname{ch}\alpha x + \operatorname{sh}\alpha x$ and $e^{-\alpha x} = \operatorname{ch}\alpha x - \operatorname{sh}\alpha x$,

the equation of elastic line takes the form:

$$y_0(x) = (\operatorname{ch}\alpha x - \operatorname{sh}\alpha x)\left[B_1 \cos\alpha x + B_2 \sin\alpha x\right] + (\operatorname{ch}\alpha x + \operatorname{sh}\alpha x)\left[B_3 \cos\alpha x + B_4 \sin\alpha x\right],$$

and after regrouping of the members of the expression:

$$y_0(x) = \operatorname{ch}\alpha x\left[\cos\alpha x\underbrace{(B_1 + B_3)}_{C_1} + \sin\alpha x\underbrace{(B_2 + B_4)}_{C_2}\right] + \operatorname{sh}\alpha x\left[\cos\alpha x\underbrace{(-B_1 + B_3)}_{C_3} + \sin\alpha x\underbrace{(-B_2 + B_4)}_{C_4}\right]$$

By introducing the new constants C_1-C_4 the final solution reads:

$$y_0(x) = \operatorname{ch}\alpha x\left[C_1 \cos\alpha x + C_2 \sin\alpha x\right] + \operatorname{sh}\alpha x\left[C_3 \cos\alpha x + C_4 \sin\alpha x\right].$$

1.1 Derivation of the integration constants C_1-C_4

By differentiation of the above equation we get:

$$y_0'(x) = \alpha\cdot\operatorname{sh}\alpha x\left[C_1\cos\alpha x + C_2\sin\alpha x\right] + \operatorname{ch}\alpha x\cdot\alpha\cdot\left[-C_1\sin\alpha x + C_2\cos\alpha x\right] +$$
$$+\alpha\cdot\operatorname{ch}\alpha x\left[C_3\cos\alpha x + C_4\sin\alpha x\right] + \operatorname{sh}\alpha x\cdot\alpha\cdot\left[-C_3\sin\alpha x + C_4\cos\alpha x\right].$$

After regrouping of the members about integration constants C_i the first derivative takes the form:

$$y_0'(x) = \alpha\cdot C_1\left(\operatorname{sh}\alpha x\cdot\cos\alpha x - \operatorname{ch}\alpha x\cdot\sin\alpha x\right) + \alpha\cdot C_2\left(\operatorname{sh}\alpha x\cdot\sin\alpha x + \operatorname{ch}\alpha x\cdot\cos\alpha x\right) +$$
$$+\alpha\cdot C_3\left(\operatorname{ch}\alpha x\cdot\cos\alpha x - \operatorname{sh}\alpha x\cdot\sin\alpha x\right) + \alpha\cdot C_4\left(\operatorname{ch}\alpha x\cdot\sin\alpha x + \operatorname{sh}\alpha x\cdot\cos\alpha x\right).$$

By differentiation of the first derivative $y_0'(x)$ the second derivative of the elastic line is:

$$y_0''(x) = \alpha^2\cdot C_1\left(\operatorname{ch}\alpha x\cdot\cos\alpha x - \operatorname{sh}\alpha x\cdot\sin\alpha x - \operatorname{sh}\alpha x\cdot\sin\alpha x - \operatorname{ch}\alpha x\cdot\cos\alpha x\right) +$$
$$+\alpha^2\cdot C_2\left(\operatorname{ch}\alpha x\cdot\sin\alpha x + \operatorname{sh}\alpha x\cdot\cos\alpha x + \operatorname{sh}\alpha x\cdot\cos\alpha x - \operatorname{ch}\alpha x\cdot\sin\alpha x\right) +$$
$$+\alpha^2\cdot C_3\left(\operatorname{sh}\alpha x\cdot\cos\alpha x - \operatorname{ch}\alpha x\cdot\sin\alpha x - \operatorname{ch}\alpha x\cdot\sin\alpha x - \operatorname{sh}\alpha x\cdot\cos\alpha x\right) +$$
$$+\alpha^2\cdot C_4\left(\operatorname{sh}\alpha x\cdot\sin\alpha x + \operatorname{ch}\alpha x\cdot\cos\alpha x + \operatorname{ch}\alpha x\cdot\cos\alpha x - \operatorname{sh}\alpha x\cdot\sin\alpha x\right).$$

Finally, for the second derivative we have:

$$y_0''(x) = -2\alpha^2\cdot C_1\cdot\operatorname{sh}\alpha x\cdot\sin\alpha x + 2\alpha^2\cdot C_2\cdot\operatorname{sh}\alpha x\cdot\cos\alpha x - 2\alpha^2\cdot C_3\cdot\operatorname{ch}\alpha x\cdot\sin\alpha x +$$
$$+2\alpha^2\cdot C_4\cdot\operatorname{ch}\alpha x\cdot\cos\alpha x.$$

After differentiation of the second derivative the third derivative of $y_0(x)$ becomes:

$$y_0'''(x) = -2\alpha^3\cdot C_1\left(\operatorname{ch}\alpha x\cdot\sin\alpha x + \operatorname{sh}\alpha x\cdot\cos\alpha x\right) + 2\alpha^3\cdot C_2\left(\operatorname{ch}\alpha x\cdot\cos\alpha x - \operatorname{sh}\alpha x\cdot\sin\alpha x\right) -$$
$$-2\alpha^3\cdot C_3\left(\operatorname{sh}\alpha x\cdot\sin\alpha x + \operatorname{ch}\alpha x\cdot\cos\alpha x\right) + 2\alpha^3\cdot C_4\left(\operatorname{sh}\alpha x\cdot\cos\alpha x - \operatorname{ch}\alpha x\cdot\sin\alpha x\right).$$

Knowing that $\frac{dy}{dx}=\varphi(x)$, $\frac{d^2y}{dx^2}=-\frac{M}{EI}$ and $\frac{d^3y}{dx^3}=-\frac{Q}{EI}$ we can obtain the general expressions for the slope of the deflected line $\varphi(x)$, for the bending moment $M(x)$ and for the shear force $Q(x)$ at any point of distance x at the beam axis. Taking in these equations $x=0$, bearing in mind that $\sin 0=0$, $\operatorname{sh} 0=0$, $\cos 0=1$, $\operatorname{ch} 0=1$ and $\cos 0\cdot\operatorname{ch} 0=1$, we get the initial parameters of the left end of the beam as follows:

$$y_0(0)=y_0=C_1;$$

$$y_0'(0)=\varphi_0=\alpha\cdot C_2+\alpha\cdot C_3;$$

$$y_0''(x)=-\frac{M_0}{EI}=2\alpha^2\cdot C_4;$$

$$y_0'''(x)=-\frac{Q_0}{EI}=2\alpha^3\cdot C_2-2\alpha^3\cdot C_3.$$

After simple transformations:

$$y_0=C_1;$$

$$\frac{\varphi_0}{\alpha}=C_2+C_3;$$

$$-\frac{M_0}{2\alpha^2\cdot EI}=C_4;$$

$$-\frac{Q_0}{2\alpha^3\cdot EI}=C_2-C_3.$$

Now expressing the constants C_1-C_4 as unknowns, from the above system of equations we have:

$$C_1=y_0;$$

$$C_2=\frac{\varphi_0}{2\alpha}-\frac{Q_0}{4\alpha^3\cdot EI};$$

$$C_3=\frac{\varphi_0}{2\alpha}+\frac{Q_0}{4\alpha^3\cdot EI}$$

$$C_4=-\frac{M_0}{2\alpha^2\cdot EI}.$$

Substituting these results in the above expression for the solution of homogeneous differential equation $y_0(x)$ we get:

$$y_0(x)=\operatorname{ch}\alpha x\left[y_0\cos\alpha x+\left(\frac{\varphi_0}{2\alpha}-\frac{Q_0}{4\alpha^3\cdot EI}\right)\sin\alpha x\right]+\operatorname{sh}\alpha x\left[\left(\frac{\varphi_0}{2\alpha}+\frac{Q_0}{4\alpha^3\cdot EI}\right)\cos\alpha x-\frac{M_0}{2\alpha^2\cdot EI}\sin\alpha x\right].$$

After regrouping of the members about the initial parameters the solution becomes:

$$y_0(x)=\operatorname{ch}\alpha x\cdot\cos\alpha x\cdot y_0+\frac{(\operatorname{ch}\alpha x\cdot\sin\alpha x+\operatorname{sh}\alpha x\cdot\cos\alpha x)}{2}\frac{\varphi_0}{\alpha}+$$

$$+\frac{(\operatorname{ch}\alpha x\cdot\sin\alpha x-\operatorname{sh}\alpha x\cdot\cos\alpha x)}{4}\left(-\frac{Q_0}{\alpha^3\cdot EI}\right)+\frac{\operatorname{sh}\alpha x\cdot\sin\alpha x}{2}\left(-\frac{M_0}{\alpha^2\cdot EI}\right).$$

By introducing the new functions $A(\alpha x)$, $B(\alpha x)$, $C(\alpha x)$ and $D(\alpha x)$ in such a way that the following substitutions are valid:

$A(\alpha x) = \text{ch}\,\alpha x \cdot \cos\alpha x$,

$B(\alpha x) = \dfrac{(\text{ch}\,\alpha x \cdot \sin\alpha x + \text{sh}\,\alpha x \cdot \cos\alpha x)}{2}$,

$C(\alpha x) = \dfrac{\text{sh}\,\alpha x \cdot \sin\alpha x}{2}$ and

$D(\alpha x) = \dfrac{(\text{ch}\,\alpha x \cdot \sin\alpha x - \text{sh}\,\alpha x \cdot \cos\alpha x)}{4}$,

the equation of elastic line becomes:

$$y_0(x) = A(\alpha x)\cdot y_0 + B(\alpha x)\frac{\varphi_0}{\alpha} - C(\alpha x)\frac{M_0}{\alpha^2 \cdot EI} - D(\alpha x)\frac{Q_0}{\alpha^3 \cdot EI}.$$

The functions $A(\alpha x)$, $B(\alpha x)$, $C(\alpha x)$ and $D(\alpha x)$ are known as Krilov's functions, and the following expressions are true for their first derivatives:

$$A'(\alpha x) = -4\alpha \cdot D(\alpha x);\quad B'(\alpha x) = \alpha \cdot A(\alpha x);\quad C'(\alpha x) = \alpha \cdot B(\alpha x);\quad D'(\alpha x) = \alpha \cdot C(\alpha x).$$

Recall that the solution of original differential equation is:

$y(x) = y_0(x) + v(x)$,

where $v(x)$ is a particular integral corresponding to the applied loads.

In that respect the equation of elastic line of a beam on elastic foundation get the form:

$$y(x) = A(\alpha x)\cdot y_0 + B(\alpha x)\frac{\varphi_0}{\alpha} - C(\alpha x)\frac{M_0}{\alpha^2 \cdot EI} - D(\alpha x)\frac{Q_0}{\alpha^3 \cdot EI} + v(x)$$

Using the above relations for first derivative of the Krilov's functions and final equation of elastic line we can obtain general expressions for the slope of the deflected line $\varphi(x)$, for the bending moment $M(x)$ and for the shear force $Q(x)$ at any point of distance x of the beam axis. These relationships are as follows:

$$y(x) = A(\alpha x)\cdot y_0 + B(\alpha x)\frac{\varphi_0}{\alpha} - C(\alpha x)\frac{M_0}{\alpha^2 \cdot EI} - D(\alpha x)\frac{Q_0}{\alpha^3 \cdot EI} + v(x),$$

$$\varphi(x) = y'(x) = -4\alpha \cdot D(\alpha x)\cdot y_0 + \alpha \cdot A(\alpha x)\frac{\varphi_0}{\alpha} - \alpha \cdot B(\alpha x)\frac{M_0}{\alpha^2 \cdot EI} - \alpha \cdot C(\alpha x)\frac{Q_0}{\alpha^3 \cdot EI} + v'(x),$$

$$M(x) = -EIy''(x) = EI \cdot 4\alpha^2 C(\alpha x)\cdot y_0 + EI \cdot 4\alpha^2 D(\alpha x)\frac{\varphi_0}{\alpha} + EI \cdot \alpha^2 A(\alpha x)\frac{M_0}{\alpha^2 \cdot EI}$$
$$+ EI \cdot \alpha^2 B(\alpha x)\frac{Q_0}{\alpha^3 \cdot EI} - EI \cdot v''(x),$$

$$Q(x) = -EIy'''(x) = EI \cdot 4\alpha^3 B(\alpha x)\cdot y_0 + EI \cdot 4\alpha^3 C(\alpha x)\frac{\varphi_0}{\alpha} - EI \cdot 4\alpha^3 D(\alpha x)\frac{M_0}{\alpha^2 \cdot EI}$$
$$+ EI \cdot \alpha^3 A(\alpha x)\frac{Q_0}{\alpha^3 \cdot EI} - EI \cdot v'''(x).$$

The same equations written in matrix form are:

$$\begin{Bmatrix} y(x) \\ \varphi(x) \\ M(x) \\ Q(x) \end{Bmatrix} = \begin{bmatrix} A & B & -C & -D \\ -4\alpha\cdot D & \alpha\cdot A & -\alpha\cdot B & -\alpha\cdot C \\ EI\cdot 4\alpha^2 C & EI\cdot 4\alpha^2 D & EI\cdot\alpha^2 A & EI\cdot\alpha^2 B \\ EI\cdot 4\alpha^3 B & EI\cdot 4\alpha^3 C & -EI\cdot 4\alpha^3 D & EI\cdot\alpha^3 A \end{bmatrix} \begin{Bmatrix} y_0 \\ \dfrac{\varphi_0}{\alpha} \\ \dfrac{M_0}{\alpha^2\cdot EI} \\ \dfrac{Q_0}{\alpha^3\cdot EI} \end{Bmatrix} + \begin{Bmatrix} v(x) \\ v'(x) \\ -EI\cdot v''(x) \\ -EI\cdot v'''(x) \end{Bmatrix}$$

In the above expressions the initial integral constants C_1-C_4 are replaced by the y_0, φ_0, M_0 and Q_0 quantities, called initial parameters. This representation is known as the method of initial conditions.

It is more convenient to express EI multiple values of the transverse displacements ($EIy(x)$) and EI multiple values of slope of deflection line ($EI\varphi(x)$). If the following relations are valid:

$\mathrm{V} = EIy(x), \quad \mathrm{V}_0 = EIy_0$;

$\Phi = EI\varphi(x), \quad \Phi_0 = EI\varphi_0$;

$\bar{\mathrm{V}} = EIv(x), \quad \bar{\Phi} = EIv'(x), \quad \bar{\mathrm{M}} = -EIv''(x), \quad \bar{\mathrm{Q}} = -EIv'''(x)$;

The basic unknowns of any section of the beam axis expressed by the initial parameters finally get the form:

$$\begin{Bmatrix} \mathrm{V}(x) \\ \Phi(x) \\ M(x) \\ Q(x) \end{Bmatrix} = \begin{bmatrix} A & \dfrac{B}{\alpha} & -\dfrac{C}{\alpha^2} & -\dfrac{D}{\alpha^3} \\ -4\alpha\cdot D & A & -\dfrac{B}{\alpha} & -\dfrac{C}{\alpha^2} \\ 4\alpha^2 C & 4\alpha\ D & A & \dfrac{B}{\alpha} \\ 4\alpha^3 B & 4\alpha^2 C & -4\alpha\ D & A \end{bmatrix} \begin{Bmatrix} \mathrm{V}_0 \\ \Phi_0 \\ M_0 \\ Q_0 \end{Bmatrix} + \begin{Bmatrix} \bar{\mathrm{V}} \\ \bar{\Phi} \\ \bar{\mathrm{M}} \\ \bar{\mathrm{Q}} \end{Bmatrix}$$

III. Derivation of particular integrals corresponding to concentrated moment M, concentrated vertical force F and uniformly distributed load q(x)

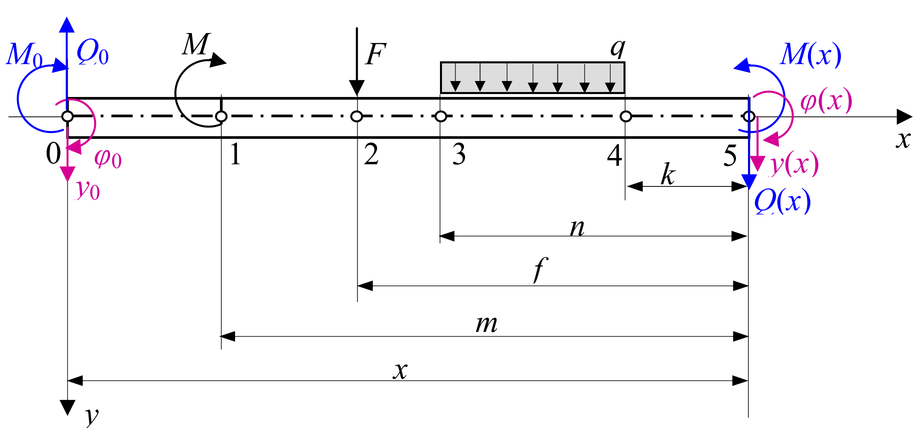

Figure 3 Beam on elastic foundation – different types of loading

Let us assume that initial parameters y_0, φ_0, M_0 and Q_0 are known. Then we can proceed from the left end of the beam (Fig. 3) toward the right along the unloaded portion 0-1 until we arrive at the point 1 where the concentrated moment is applied.

1. Particular integral owe to the concentrated moment *M*

The concentrated moment *M* must have an effect to the right of section 1 similar to the effect, which the initial moment M_0 had on portion 0-1. The influence of M_0 is given by the third column of the above matrix. In accordance with this column the influence of concentrated moment can be expressed as:

$$\bar{\mathrm{V}}(x) = -\frac{C(\alpha m)}{\alpha^2} M,$$

$$\bar{\Phi}(x) = -\frac{B(\alpha m)}{\alpha} M,$$

$$\bar{\mathrm{M}} = A(\alpha m) \cdot M,$$

$$\bar{\mathrm{Q}} = -4\alpha \; D(\alpha m) \cdot M.$$

2. Particular integral due to the concentrated force *F*

In a similar way we can find the influence of concentrated force *F*. It is the same as the influence of Q_0 to the portion 0-2 of the beam, taken with reverse sign (because concentrated force is opposing to the initial shear force). Thus, the influence of *F* could be obtained by the forth column of the above matrix, or:

$$\bar{\mathrm{V}}(x) = \frac{D(\alpha f)}{\alpha^3} F,$$

$$\bar{\Phi}(x) = \frac{C(\alpha f)}{\alpha^2} F,$$

$$\bar{\mathrm{M}}(x) = -\frac{B(\alpha f)}{\alpha} F,$$

$$\bar{\mathrm{Q}}(x) = -A(\alpha f) \cdot F.$$

2. Particular integral corresponding to the distributed load *q*

The distributed load q can be regarded as consisting of infinite small concentrated forces such as $q \cdot dx_q$ in Fig. 4. The effect of this force for the portion 3-5 of the beam into consideration is the same as the effect of the force F (Fig. 4), namely:

$$\bar{\mathrm{V}}(x) = \frac{D\left(\alpha(n - x_q)\right)}{\alpha^3} q \cdot dx_q .$$

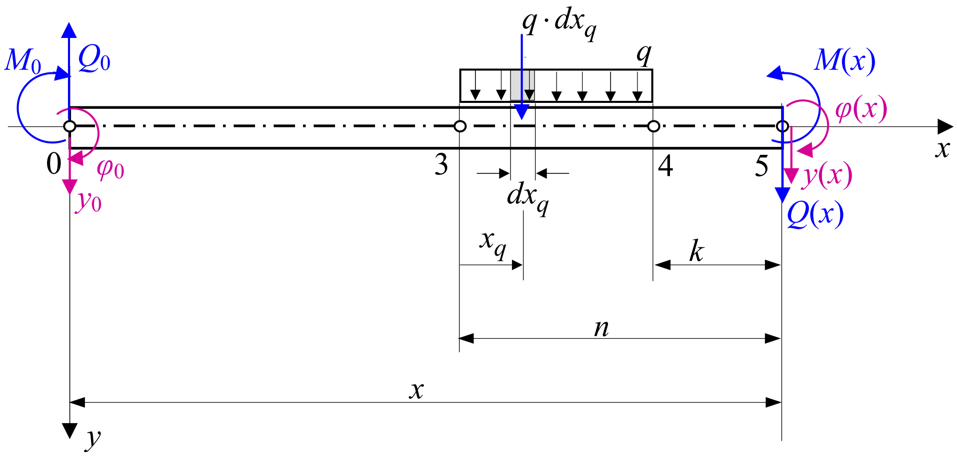

Figure 4 Derivation of particular integral due to distributed loading

The effect of all infinite small concentrated forces belonging to the distributed load can be expressed as the following integral:

$$\bar{V}(x) = \int_0^{n-k} \frac{D\left(\alpha(n-x_q)\right)}{\alpha^3} q \cdot dx_q = -q \int_0^{n-k} \frac{D\left(\alpha(n-x_q)\right)}{\alpha^3} \cdot d\left(n-x_q\right)$$

Bearing in mind that $A'(\alpha x) = -4\alpha \cdot D(\alpha x)$ it can be written that:

$\int A'(\alpha x) \cdot dx = -4\alpha \cdot \int D(\alpha x) \cdot dx$,

respectively $\int D(\alpha x) \cdot dx = -A(\alpha x)/4\alpha$.

In our case $\int D\left(\alpha(n-x_q)\right) \cdot d(n-x_q) = -A\left(\alpha(n-x_q)\right)/4\alpha$.

Finally, for the influence of distributed load on the transverse displacements we get:

$$\bar{V}(x) = -\frac{q}{\alpha^3} \frac{-A\left(\alpha(n-x_q)\right)}{4\alpha} \Bigg|_0^{n-k} = -\frac{q}{\alpha^3} \frac{-A\left(\alpha(n-n+k)\right) + A\left(\alpha(n-0)\right)}{4\alpha} = -\frac{q}{4\alpha^4}\left(A(\alpha n) - A(\alpha k)\right)$$

$$\bar{V}(x) = -\frac{q}{4\alpha^4}\left(A(\alpha n) - A(\alpha k)\right).$$

The other particular integrals can be derived by differentiation of the above equation - recall the relations $A' = -4\alpha \cdot D$; $B' = \alpha \cdot A$; $C' = \alpha \cdot B$; $D' = \alpha \cdot C$, or:

$$\bar{\Phi}(x) = \bar{V}'(x) = \frac{q}{\alpha^3}\left(D(\alpha n) - D(\alpha k)\right);$$

$$\bar{M}(x) = -\bar{\Phi}'(x) = -\frac{q}{\alpha^2}\left(C(\alpha n) - C(\alpha k)\right);$$

$$\bar{Q}(x) = \bar{M}'(x) = -\frac{q}{\alpha}\left(B(\alpha n) - B(\alpha k)\right).$$

IV. Classification of the beams according to their stiffness

The term $\alpha \cdot l$, where l is the beam length, characterizes the relative stiffness of the beam on elastic foundation. According to the values of $\alpha \cdot l$ the beams can be classified into three groups:

1) Short beams for which: $\alpha \cdot l < 0.5$;
2) Beams of medium length: $0.5 \geq \alpha \cdot l \leq 5$;
3) Long beams: $\alpha \cdot l > 5$.

For beams belonging to the first group the bending deformations of the beam can be neglected in the most practical cases. These deformations are negligible small compared to the deformations produced in the foundation. Therefore such beams can be considered as absolutely rigid.
For the beams belonging to the second group the loads applied at the one end of the beam have a finite and not negligible effect to the other end. In this case an accurate computation of the beam is necessary and no approximations are possible. For these beams the method of initial conditions is very suitable and the obtained results are accurate.
The beams belonging to the third group have a specific feature $\alpha \cdot l$ such that the counter effect of the end conditioning terms (forces and displacements) on each other is negligible. When investigate the one end of the beam we can assume that the other end is infinitely far away. The forces applied at the one end of the beam have a negligible effect to the other end. In other words the Krilov's functions $A(\alpha l)$, $B(\alpha l)$, $C(\alpha l)$ and $D(\alpha l)$ are practically zero which greatly simplifies the computations.

V. Numerical example

In the following numerical example we shall construct the diagrams of vertical displacements, the slope of the deflection line, bending moment and shear force and the diagram of vertical reactions in the foundation. All the diagrams will be found by the method of initial conditions.
Consider a beam on elastic foundation with free ends. The geometrical dimensions, mechanical properties and loadings are shown in Fig. 5. The modulus of elasticity of material of the beam is $3 \cdot 10^7$ kN/m^2 (concrete) and the modulus of the foundation is k_0=50000 kN/m^2/m.
The Wikler's constant or constant of the foundation is:
$k = k_0 \cdot b = 50000 \cdot 1.1 = 55000$.

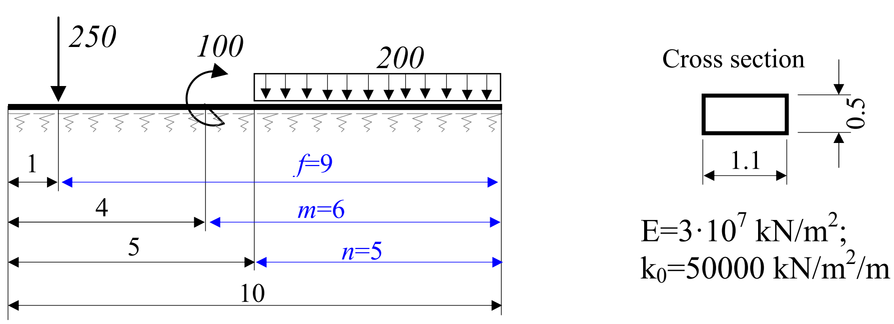

Figure 5 Numerical example: geometrical dimensions and mechanical properties

The main characteristics of the beam into consideration are:

$$I = \frac{1}{12} b \cdot h^3 = \frac{1}{12} 1.1 \cdot 0.5^3 = 0.011458\, m^4;$$

$$EI = 3 \cdot 10^7 \cdot 0.011458 = 343750\, kN.m^2;$$

$$\alpha = \sqrt[4]{\frac{k}{4EI}} = \sqrt[4]{\frac{55000}{4 \cdot 343750}} = 0.44721\, m^{-1};$$

$$\alpha \cdot l = 0.44721 \cdot 10 = 4.472.$$

The beam is of a medium length according to its stiffness $\alpha \cdot l$, so the method of initial conditions is applicable.

Next we should determine the initial parameters of the left end using the boundary conditions of the right end of the beam. Obviously for the free left end the bending moment M_0 and the shear force Q_0 are equal to zero, the vertical deflection y_0 and rotation φ_0 are the unknown initial parameters which should be determined using the boundary conditions of the right end. These boundary conditions are:

$M(l) = 0;$

$Q(l) = 0.$

Using the expressions for bending moment and shear force in terms of initial parameters and accounting for the influence of loading the following equations are compounded:

$$M(l) = 4\alpha^2 C(\alpha l) \cdot V_0 + 4\alpha\ D(\alpha l) \cdot \Phi_0 + A(\alpha m) \cdot M - \frac{B(\alpha f)}{\alpha} F - \frac{q}{\alpha^2}\left(C(\alpha n) - C(\alpha k)\right) = 0,$$

$$Q(l) = 4\alpha^3 B(\alpha l) \cdot V_0 + 4\alpha^2 C(\alpha l) \cdot \Phi_0 - 4\alpha\ D(\alpha m) \cdot M - A(\alpha f) \cdot F - \frac{q}{\alpha}\left(B(\alpha n) - B(\alpha k)\right) = 0.$$

For $x=l$ the distances m, f, n and k are as shown in Fig. 5 or m =6, f =9, n =5 and k =0. In this case the above equations become:

$$-17.004 \cdot V_0 - 14.358 \cdot \Phi_0 - 6.5921 \cdot 100 + 44.042 \cdot 250 - 1000(1.81928 - 0) = 0,$$

$$-9.4692 \cdot V_0 - 17.004 \cdot \Phi_0 - 4.3782 \cdot 100 + 17.7663 \cdot 250 - 447.21(0.43394 - 0) = 0.$$

$$-17.004 \cdot V_0 - 14.358 \cdot \Phi_0 = -8532.07,$$

$$-9.4691 \cdot V_0 - 17.004 \cdot \Phi_0 = -3810.01.$$

Wherefrom:

$V_0 = 590.03$

$\Phi_0 = -104.51$

Having the initial parameters available, for the given external loadings, we can obtain any parameter of an arbitrary section of the beam axis using the expressions below:

$$\begin{Bmatrix} V(x) \\ \Phi(x) \\ M(x) \\ Q(x) \end{Bmatrix} = \begin{bmatrix} A(\alpha x) & \dfrac{B(\alpha x)}{\alpha} & -\dfrac{C(\alpha x)}{\alpha^2} & -\dfrac{D(\alpha x)}{\alpha^3} \\ -4\alpha \cdot D(\alpha x) & A(\alpha x) & -\dfrac{B(\alpha x)}{\alpha} & -\dfrac{C(\alpha x)}{\alpha^2} \\ 4\alpha^2 C(\alpha x) & 4\alpha \; D(\alpha x) & A(\alpha x) & \dfrac{B(\alpha x)}{\alpha} \\ 4\alpha^3 B(\alpha x) & 4\alpha^2 C(\alpha x) & -4\alpha \; D(\alpha x) & A(\alpha x) \end{bmatrix} \begin{Bmatrix} V_0 \\ \Phi_0 \\ M_0 \\ Q_0 \end{Bmatrix} + \begin{Bmatrix} \overline{V} \\ \overline{\Phi} \\ \overline{M} \\ \overline{Q} \end{Bmatrix}$$

$$\begin{Bmatrix} \overline{V} \\ \overline{\Phi} \\ \overline{M} \\ \overline{Q} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{C(\alpha m)}{\alpha^2} M \\ -\dfrac{B(\alpha m)}{\alpha} M \\ A(\alpha m) \cdot M \\ -4\alpha \; D(\alpha m) \cdot M \end{Bmatrix} + \begin{Bmatrix} \dfrac{D(\alpha f)}{\alpha^3} F \\ \dfrac{C(\alpha f)}{\alpha^2} F \\ -\dfrac{B(\alpha f)}{\alpha} F \\ -A(\alpha f) \cdot F \end{Bmatrix} + \begin{Bmatrix} -\dfrac{q}{4\alpha^4}\left(A(\alpha n) - A(\alpha k)\right) \\ \dfrac{q}{\alpha^3}\left(D(\alpha n) - D(\alpha k)\right) \\ -\dfrac{q}{\alpha^2}\left(C(\alpha n) - C(\alpha k)\right) \\ -\dfrac{q}{\alpha}\left(B(\alpha n) - B(\alpha k)\right) \end{Bmatrix}$$

$$R(x) = k \cdot y(x) = k \cdot \frac{V(x)}{EI}$$

It should be pointed out that for every different section of the beam, with a single abscissa x, the values of m, f, n and k are different and depend on the distance x. The influence of different loading appears when the section into consideration is on the right of this loading. The obtained results for vertical displacements, the slope of the deflection line, bending moments, shear forces and the vertical reactions, for different sections of the beam, are calculated and given in table 1.

Table 1

x	***V*(*x*)=EI*y*(*x*)**	***Φ*(*x*)=EI*φ*(*x*)**	***M*(*x*)**	***Q*(*x*)**	***R*(*x*)**
0	590.035	-104.514	0.000	0.000	94.406
0.9999	481.727	-119.549	44.395	85.922	77.076
1	481.727	-119.549	44.395	-164.078	77.076
2	364.279	-93.929	-84.426	-96.754	58.285
3	326.363	29.653	-153.863	-43.162	52.218
3.9999	437.891	195.983	-169.054	15.748	70.063
4	437.891	195.983	-69.054	15.748	70.063
5	662.582	244.093	-12.626	103.125	106.013
6	899.391	219.197	49.981	28.431	143.903
7	1090.916	162.960	55.849	-11.586	174.547
8	1228.738	116.131	35.530	-25.386	196.598
9	1331.329	93.128	11.337	-20.275	213.013
10	1421.503	89.150	0.000	0.000	227.440

The diagrams of required displacements and internal forces of the beam are presented in Fig. 6. The diagram of continuously distributed reaction forces in the foundation is depicted in Fig. 7.

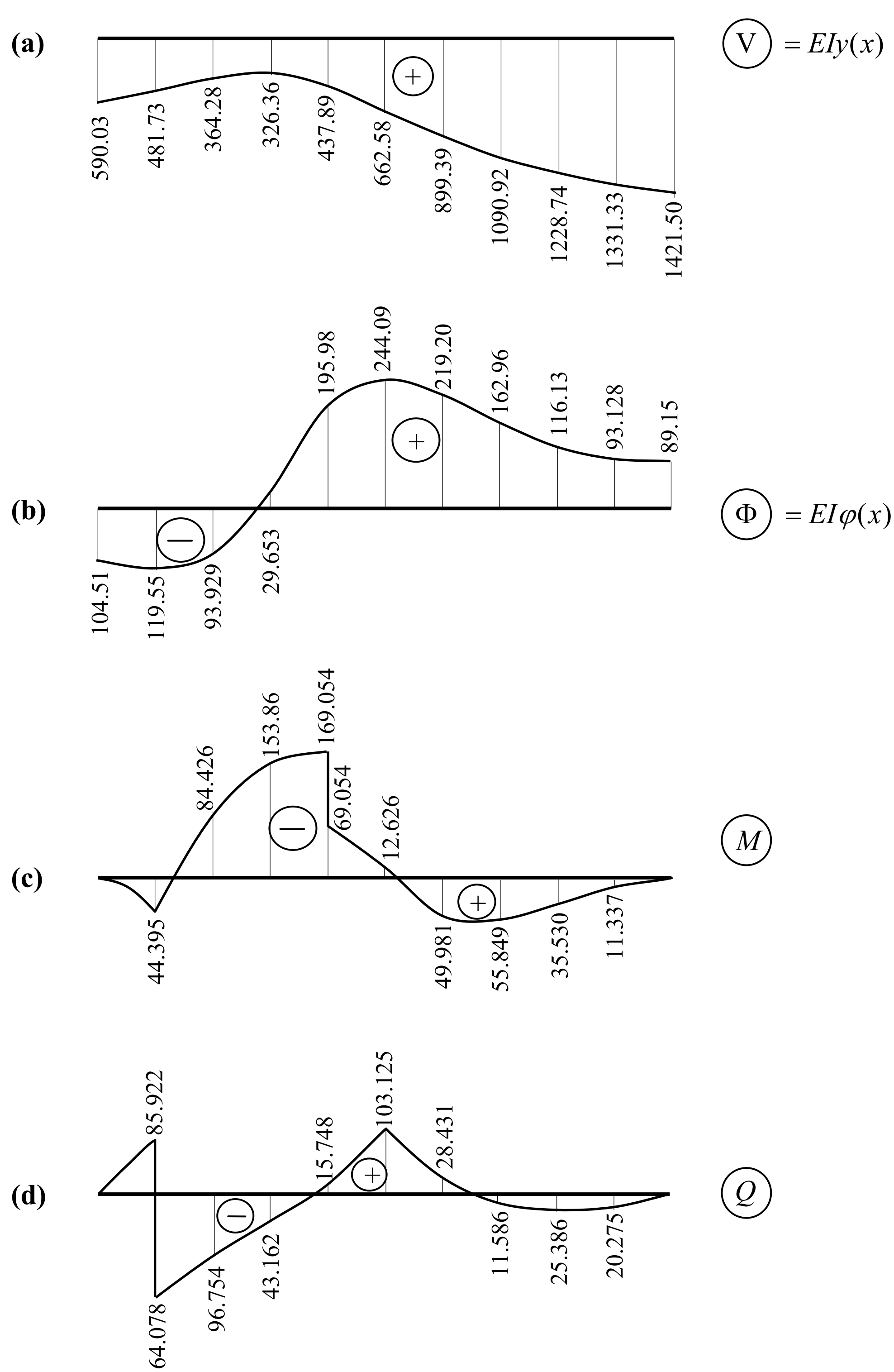

Figure 6 (a) Vertical deflections; (b) slope of the deflected line; (c) bending moment diagram; (d) shear force diagram

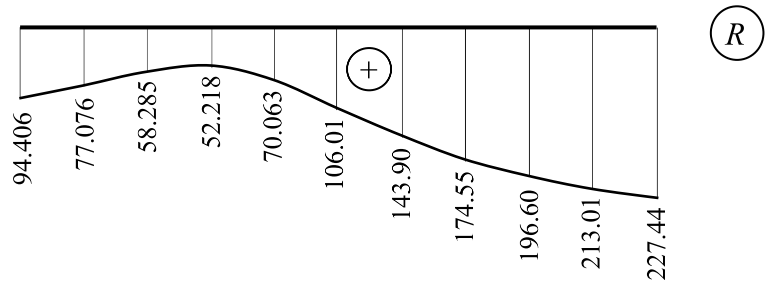

Figure 7 Diagram of vertical reactions

Verification

In order to verify the obtained results we shall check the equilibrium of the vertical forces using the condition $\Sigma V = 0$. In order to do that, we should find the resultant force of distributed vertical reaction in the foundation, presented in Fig. 7. In other words we have to calculate the area of the corresponding diagram of *R*. Using a numerical integration the area of the diagram of vertical reactions is:

$$A = \frac{2}{3}1\left(\frac{94.406}{2} + 2\cdot 77.076 + 58.285 + 2\cdot 52.218 + 70.063 + 2\cdot 106.01 + 143.90 + 2\cdot 174.55 + 196.60 + \right.$$

$$\left. + 2\cdot 213.01 + \frac{227.44}{2}\right) = 1250.33$$

$$\Sigma V = 0$$

$$1250.33 - 250 - 200\cdot 5 = 1250.33 - 1250$$

The numerical error is 0.026%. The error is due to numerical integration and will decrease if we calculate the values of vertical reactions in more sections of the beam, respectively if we decrease the step between two neighboring values of the diagram.

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