Tables of derivatives

I. Basic rules of differentiation

Suppose that there exist $u\,'(x)$ and $v\,'(x)$ for any $x \in D \subseteq \mathbb{R}$. Then for $x \in D$ hold the formulas:

1. $c\,' = 0,\ (c = const),\ x\,' = 1$;
2. $(u(x) \pm v(x))\,' = u\,'(x) \pm v\,'(x)$ – the sum rule;
3. $(c.u(x))\,' = c.u\,'(x)\ (c = const)$;
4. $(u(x).v(x))\,' = u\,'(x).v(x) + u(x).v\,'(x)$ – the product rule;
5. $\left(\dfrac{u(x)}{v(x)}\right)' = \dfrac{u\,'(x).v(x) - u(x).v\,'(x)}{v^2(x)}$ – the quotient rule;
6. If the function $u = u(x)$ is differentiable on the set D, and the function $y = f(u)$ is differentiable on the set $G = u(D)$, then the composite function $y(x) = f(u(x))$ is differentiable on the set D and holds the formula

$$y'(x) = f\,'(u)\ .u\,'(x),\ x \in D,$$

called formula for derivative of a **composition of functions**, or **the chain rule**.

II. Differential of function. Properties

Let exists $y\,'(x),\ x \in D$. The expression

$$d\,y(x) = y\,'(x).dx,\ \text{where } dx = \Delta x = h \neq 0$$

is the variation of the independent variable x, (also called **differential of the independent variable**), is called **first differential** of y. In this denotations the derivative can be written as a quotient of differentials of y and x (denotation of Leibniz):

$$y\,'(x) = \frac{dy}{dx}.$$

Differential possesses the following properties:

1. $dc = c'.dx = 0,\ (c = const)$;
2. $d(u(x) \pm v(x)) = (u(x) \pm v(x))\,'.dx = u\,'(x).dx \pm v\,'(x).dx = du(x) \pm dv(x)$;
3. $d(c.u(x))\,' = (c.u\,'(x)).dx = c.du(x)\ (c = const)$;
4. $d(a.u(x) + c) = (a.u(x) + c)'.dx = a.u'(x)dx = a.du(x)$, where a and c are arbitrary constants;
5. $d(u(x).v(x)) = (u(x).v(x))\,'.dx = (u\,'(x).v(x) + u(x).v\,'(x)).dx = v(x).du(x) + u(x).dv(x)$;
6. $d\left(\dfrac{u(x)}{v(x)}\right) = \left(\dfrac{u\,'(x).v(x) - u(x).v\,'(x)}{v^2(x)}\right)dx = \dfrac{v(x).du(x) - u(x).dv(x)}{v^2(x)}$;

7. The differential of composite function $y(x) = f(u(x))$ is

$$dy(x) = (f(u(x)))'.dx = f'(u)\ .u'(x).dx = f'(u(x))\ .du(x).$$

III. Derivative of a function represented parametrically

Let $\varphi(t)$ and $\psi(t)$ be differentiable functions, defined on (α, β), and $\varphi'(t) \neq 0$ on this interval. If the system of equations

$$x = \varphi(t),\ y = \psi(t),\ t \in (\alpha, \beta)$$

defines y as a single-valued continuous function of x, then there exists a derivative $y'(x)$ and

$$y'(x) = \frac{dy}{dx} = \frac{y'(t).dt}{x'(t).dt} = \frac{y'(t)}{x'(t)},\ t \in (\alpha, \beta).$$

IV. Differentiation of basic elementary functions

1. $(e^x)' = e^x,\ x \in \mathbb{R}$;
2. $(\ln|x|)' = \frac{1}{x},\ x \neq 0;$
3. $(x^\alpha)' = \alpha.x^{\alpha-1},\ \alpha = const;$
4. $(a^x)' = a^x.\ln a,\ x \in \mathbb{R},\ a = const,\ a > 0,\ a \neq 1;$
5. $(\log_a|x|)' = \frac{1}{x.\ln a},\ x \neq 0;$
6. $(\cos x)' = -\sin x,\ x \in \mathbb{R};$
7. $(\sin x)' = \cos x,\ x \in \mathbb{R};$
8. $(\mathrm{tg}\, x)' = \frac{1}{\cos^2 x},\ x \neq (2k+1)\frac{\pi}{2},\ k = 0, \pm1, \pm2, \ldots;$
9. $(\mathrm{ctg}\ x)' = -\frac{1}{\sin^2 x},\ x \neq k\pi,\ k = 0, \pm1, \pm2, \ldots;$
10. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}},\ x \in (-1,\ 1);$
11. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}},\ x \in (-1,\ 1);$
12. $(\mathrm{arctg}\, x)' = \frac{1}{1+x^2},\ x \in \mathbb{R};$
13. $(\mathrm{arcctg}\, x)' = -\frac{1}{1+x^2},\ x \in \mathbb{R}.$

V. Differentiation of basic elementary functions in case that x is replaced with $u(x)$ where $u(x)$ is an arbitrary differentiable function on it's domain.

1. $(e^{u(x)})' = e^{u(x)}.u'(x)$;

2. $(\ln|u(x)|)' = \dfrac{1}{u(x)}.u'(x),\ u(x) \neq 0,\ \left(\ln|x+\sqrt{x^2 \pm a^2}\,|\right)' = \dfrac{1}{\sqrt{x^2 \pm a^2}},\ a = const;$

3. $(u^\alpha(x))' = \alpha.u^{\alpha-1}(x).u'(x),\ \alpha = const;$

4. $(a^{u(x)})' = a^{u(x)}.u'(x).\ln a,\ x \in \mathbb{R},\ a = const,\ a > 0,\ a \neq 1;$

5. $(\log_a|u(x)|)' = \dfrac{1}{u(x).\ln a}.u'(x),\ u(x) \neq 0;$

6. $(\cos u(x))' = -\sin u(x).u'(x);$

7. $(\sin u(x))' = \cos u(x).u'(x);$

8. $(\mathrm{tg}\, u(x))' = \dfrac{1}{\cos^2 u(x)}.u'(x),\ x \neq (2k+1)\dfrac{\pi}{2},\ k = 0, \pm1, \pm2, \ldots;$

9. $(\mathrm{ctg}\, u(x))' = -\dfrac{1}{\sin^2 u(x)}.u'(x),\ u(x) \neq k\pi,\ k = 0, \pm1, \pm2, \ldots;$

10. $(\arcsin u(x))' = \dfrac{1}{\sqrt{1-u^2(x)}}.u'(x),\ u(x) \in (-1,\ 1);$

11. $(\arccos u(x))' = -\dfrac{1}{\sqrt{1-u^2(x)}}.u'(x),\ u(x) \in (-1,\ 1);$

12. $(\mathrm{arctg}\, u(x))' = \dfrac{1}{1+u^2(x)}.u'(x);$

13. $(\mathrm{arcctg}\, u(x))' = -\dfrac{1}{1+u^2(x)}.u'(x).$

VI. Logarithmic derivative. Let $u(x)$ and $v(x)$ be differentiable functions on the set D, moreover let $u(x) > 0$ on D. Then the function $F(x) = u(x)^{v(x)}$ is differentiable function on D and holds the formula

$$F'(x) = \left(u(x)^{v(x)}\right)' = \left(e^{v(x).\ln u(x)}\right)' = e^{v(x).\ln u(x)}.(v(x).\ln u(x))' =$$

$$= u(x)^{v(x)}.\left(v'(x).\ln u(x) + v(x).\frac{u'(x)}{u(x)}\right).$$

Some formulas and tables for indefinite integrals

0. **Definition and properties of the indefinite integral:**

1. $F(x) = \int f(x)\,dx + C \iff F'(x) = f(x) \iff dF(x) = f(x)\,dx;$

2. $\int \alpha f(x) + \beta g(x)\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$ for any constants α and β and functions $f(x)$ and $g(x)$;

3. $F(x) = \int f(u(x))u'(x)\,dx$ and $G(u) = \int f(u)\,du + C$ then

$$F(x) = G(u(x)) + C$$

I. Tables of integrals

I. a) Basic integrals

1. $\int 0\,dx = C = const., \int 1\,dx = x + C,\ x \in (-\infty, \infty)$;

2. $\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C,\ \alpha \neq -1;$

3. $\int \frac{1}{x}\,dx = \ln|x| + C,\ x \neq 0$ (the case $\alpha = -1$);

4. $\int e^x\,dx = e^x + C, x \in (-\infty, \infty);$

5. $\int a^x\,dx = \frac{a^x}{\ln a} + C, x \in (-\infty, \infty),\ a > 0,\ a \neq 1;$

6. $\int \cos x\,dx = \sin x + C,\ x \in (-\infty, \infty);$

7. $\int \sin x\,dx = -\cos x + C,\ x \in (-\infty, \infty);$

8. $\int \frac{1}{\cos^2 x}\,dx = \tan x + C,\ x \neq (2k+1)\frac{\pi}{2}, k = \pm 1, \pm 2, \ldots;$

9. $\int \frac{1}{\sin^2 x}\,dx = -\cot x + C,\ x \neq k\pi, k = \pm 1, \pm 2, \ldots;$

10. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C = -\arccos x + C,\ x \in (-1, 1);$

11. $\int \frac{1}{1+x^2}\,dx = \arctan x + C = -\mathrm{arccotgx} + \mathrm{C},\ \mathrm{x} \in (-\infty, \infty);$

12. $\int \frac{1}{\sqrt{x^2+a}}\,dx = \ln|x + \sqrt{x^2+a}| + C,\ x > -a,\ a = const.;$

I.b) Here we suppose that $u(x)$ is a differentiable function on it's domain.

1. $\int u'(x)\,dx = \int d\,u(x) = u(x) + C$;

2. $\int u(x)^\alpha u'(x)\,dx = \int u(x)^\alpha\,du(x) = \dfrac{u(x)^{\alpha+1}}{\alpha+1} + C,\ \alpha \neq -1;$

3. $\int \dfrac{u'(x)}{u(x)}\,dx = \int \dfrac{1}{u(x)}\,du(x) = \ln|u(x)| + C,\ u(x) \neq 0\ \ \text{(the case } \alpha = -1);$

4. $\int e^{u(x)} u'(x)\,dx = \int e^{u(x)}\,du(x) = e^{u(x)} + C;$

5. $\int a^{u(x)} u'(x)\,dx = \int a^{u(x)}\,du(x) \ = \dfrac{a^{u(x)}}{\ln a} + C;$

6. $\int \cos u(x) u'(x)\,dx = \int \cos u(x)\,du(x) = \sin u(x) + C;$

7. $\int \sin u(x) u'(x)\,dx = \int \sin u(x)\,du(x) = -\cos u(x) + C;$

8. $\int \dfrac{u'(x)}{\cos^2 u(x)}\,dx = \int \dfrac{1}{cos^2 u(x)}\,du(x) = \tan u(x) + C;$

9. $\int \dfrac{u'(x)}{\sin^2 u(x)}\,dx = \int \dfrac{1}{\sin^2 u(x)}\,du(x) = -\cot u(x) + C;$

10. $\int \dfrac{u'(x)}{\sqrt{1-u^2(x)}}\,dx = \int \dfrac{1}{\sqrt{1-u^2(x)}}\,du(x) = \arcsin u(x) + C = -\arccos u(x) + C;$

11. $\int \dfrac{u'(x)}{1+u^2(x)}\,dx = \int \dfrac{1}{1+u^2(x)}\,du(x) = \arctan u(x) + C = -\text{arccotg} u(x) + \text{C};$

12. $\int \dfrac{u'(x)}{\sqrt{u(x)^2+a}}\,dx = \int \dfrac{1}{\sqrt{u^2(x)+a}}\,du(x) = \ln|u(x) + \sqrt{u^2(x)+a}| + C$

III. Integration by parts: If $u(x)$ and $v(x)$ have derivatives on their domains then holds the formula

$$\int u(x)v'(x)\,dx = \int u(x)\,dv(x) = u(x)v(x) - \int v(x)\,du(x) = u(x)v(x) - \int v(x)u'(x)\,dx = \ldots.$$

IV. Change of variables:

If

$$F(x) = \int f(x)\,dx \text{ and } G(t) = \int f(\varphi(t))\varphi'(t)\,dt + C$$

where $x = \varphi'(t) \neq 0$ on it's domain then

$F(x) = G(\varphi^{-1}(x)) + C$ where $t = \varphi^{-1}(x)$ is an inverse function of $x = \varphi(t)$.