9. Point mode plotting with more than two images 2 hours

aim:
intersection of more than two rays with orientated images

Theory:
Application co linearity equation

9.1. Spatial Resection and Intersection in Photogrammetry

The discussed procedures for outer orientation of photogrammetric images, based on one step or two steps procedures could be enlarged to general case, when more then two images are taken and can be used for object reconstruction.

The procedure of single image space resection can be used only in cases of mono plotting. Such cases usually require additional information about the shape of the objects. They have their implementation only for image rectification or in cases of plane objects.

For two images are possible three approaches:

- independent resection for every photo without taking into account the tie points.
- simultaneous resection for all images with calculation of tie points' coordinates;
- creation of photogrammetric model with taking into account of all point or only the part of them. Orientation of the model using points with known object coordinates. This approach leads to very specific results. If all points are used for relative orientation of the model then the adjusted values of image coordinates can be obtained. If adjusted values are used for calculation of spatial coordinates then it does not matter from which photos these image coordinates are used because the results will be the same. If image coordinates are not adjusted or some image points are not included in the adjustment then the space coordinates of the pointes must be averaged or adjusted. It must be pointed that in this case in the process of outer orientation the relative orientation parameters cannot be taken into account and stay fixed. By this reason space coordinates of object points cannot influence the relative orientation parameters. As a final result the adjustment is not entirely correct.

This approach can be enlarged for the cases of more than two photos. It is made in methods of strip and block adjustment by method of models. Separate models are created and after that are transformed relatively one to another on the base of common point. It is possible to connect models by sequential attachment of the next photo to the previous one. These approaches will be discussed in more details when block adjustment is explained.

This approach is possible to be used in the case of close range photogrammetry but very often the orientation of the cameras is so different that it is not possible to create stereo model. In this case only several points in the adjacent photos are measured in both images. Tie points are not enough to calculate parameters of relative orientation. This requires all photos to be calculated together for determination of orientation parameters. Usually the colinearity conditions are used for this purpose. Such method of adjustment is called bundle adjustment.

All this pequliarity require different approach when close range images of the whole object are processed for creation the model of the whole object. In the cases of close range photogrammetry the orientation of cameras differ so much that is necessary to find good approximation of this orientation.

9.2. Initial approximation for bundle triangulation

Two problems are important here:

-approximate value for camera position;

-alternate methods for space resection and intersection.

9.2.1. Approximate values for camera position and new points

To calculate approximate values for camera position it is necessary to know approximate values of angle orientations of the camera. They can be measured in the case of close range photogrammetry when the coordinate system is local and usually is oriented relatively to the main axes of the object.

The coefficients of rotation matrix can be calculated from the coordinates of unit vectors of image coordinate system. Indeed the relation between image coordinates and space coordinates is given by transformation equation.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_{\omega\varphi\kappa} \cdot \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \cdot \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \tag{9.1}$$

where for simplification we denote $\xi_0 = \eta_0 = 0$.

If we substitute for the vectors in camera coordinate system the unit vectors we obtain:

$$\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \tag{9.2}$$

for vector ***i***,

$$\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix} \qquad (9.3)$$

for ***j*** vector and

$$\boldsymbol{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} \qquad (9.4)$$

for ***k*** vector.

If we have arbitrary oriented photos in space we can calculate rotation matrixes from unit vectors.

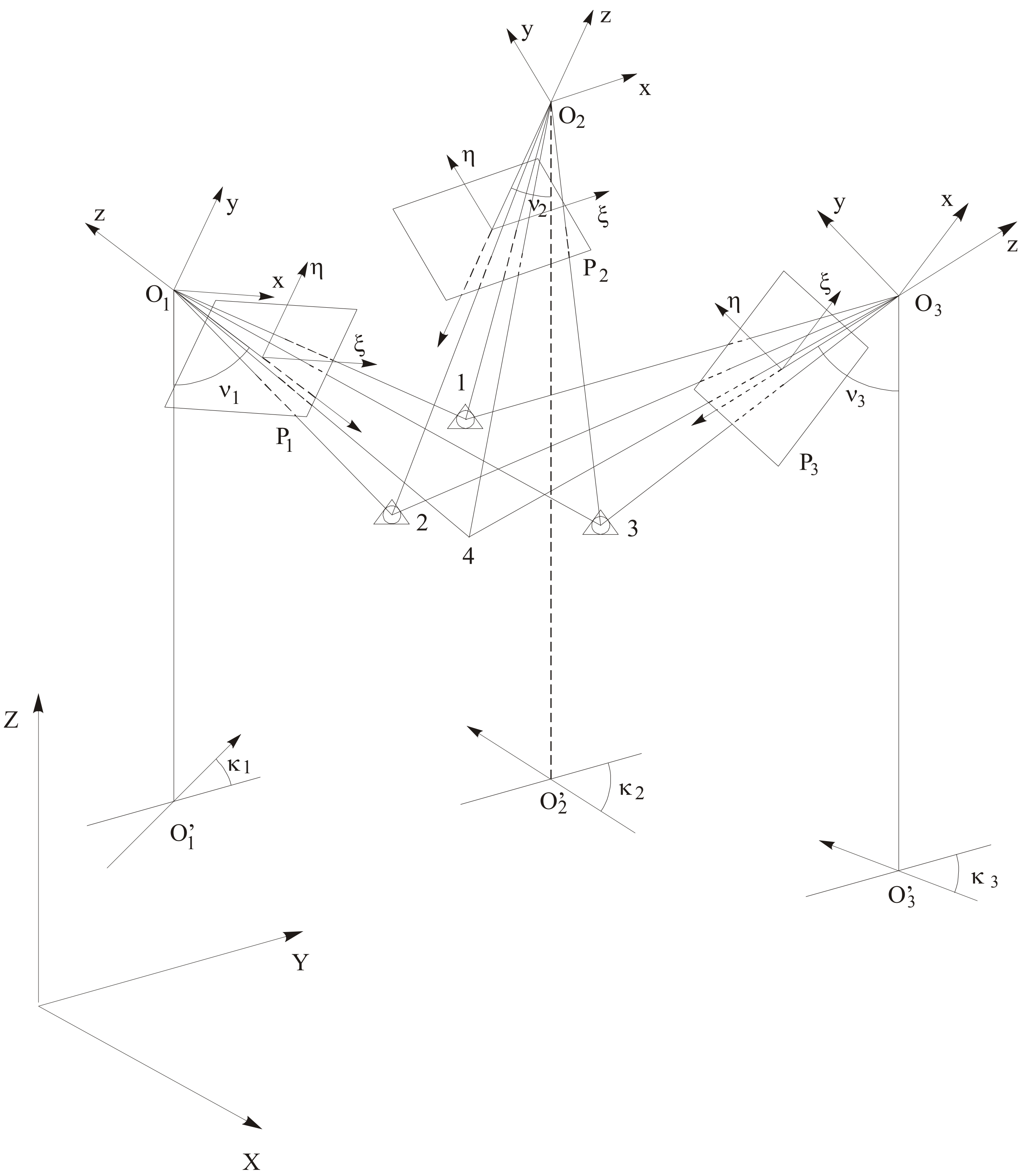


Figure 9.1. Three photographs with three control points

The calculation of elements of rotation matrixes allow to solve the initial approximation for projection centers positions and coordinates of new points from linear system of equations. From colinearity conditions making assumption for $x_0 = y_0 = 0$ we derive

$$x = -c\frac{r_{11}.(X - X_0) + r_{21}.(Y - Y_0) + r_{31}.(Z - Z_0)}{r_{13}.(X - X_0) + r_{23}.(Y - Y_0) + r_{33}.(Z - Z_0)}$$
$$y = -c\frac{r_{12}.(X - X_0) + r_{22}.(Y - Y_0) + r_{32}.(Z - Z_0)}{r_{13}.(X - X_0) + r_{23}.(Y - Y_0) + r_{33}.(Z - Z_0)} \quad (9.5)$$

We multiply the left part of equations by denominator

$$\begin{aligned} x.[r_{13}.(X - X_0) + r_{23}.(Y - Y_0) + r_{33}.(Z - Z_0)] &= \\ &= -c.[r_{11}.(X - X_0) + r_{21}.(Y - Y_0) + r_{31}.(Z - Z_0)] \\ y.[r_{13}.(X - X_0) + r_{23}.(Y - Y_0) + r_{33}.(Z - Z_0)] &= \\ &= -c.[r_{12}.(X - X_0) + r_{22}.(Y - Y_0) + r_{32}.(Z - Z_0)] \end{aligned} \quad (9.6)$$

After rearranging the terms by parameters we obtain the result

$$\begin{aligned} &(c.r_{11} + x.r_{13}).X_0 + (c.r_{21} + x.r_{23}).Y_0 + (c.r_{31} + x.r_{33}).Z_0 \\ &\quad - (c.r_{11} + x.r_{13}).X - (c.r_{21} + x.r_{23}).Y - (c.r_{31} + x.r_{33}).Z = 0 \\ &(c.r_{12} + x.r_{13}).X_0 + (c.r_{22} + x.r_{23}).Y_0 + (c.r_{32} + x.r_{33}).Z_0 \\ &\quad - (c.r_{12} + x.r_{13}).X - (c.r_{22} + x.r_{23}).Y - (c.r_{32} + x.r_{33}).Z = 0 \end{aligned} \quad (9.7)$$

In the above equations there are six unknowns – coordinates of projection center (X_0, Y_0, Z_0) and coordinate of the object point (X, Y, Z). In case when this is a control point its coordinates are given and they are added to the list of unknowns. In complicated sets of photos usually there are enough measurements to different points and it is possible to calculate all parameters. The number of points for calculation of orientation is made in Appendix 1.

$$3.k_p \le 2.n_{p1}^c + 4.n_{p2}^c + 6.n_{p3}^c + (4.n_{p2}^n - 3) + (6.n_{p3}^n - 3) + \ldots \quad (9.8)$$

where k_p is number of photos and n_p^n is the number of new points

In case when number of equations is greater then the number of required only the necessary number is taken. When the number of equations is more then required it is possible to apply least square adjustment method to find the solution.

9.2.2. Alternating resection and intersection

If the elements of outer orientation are known then it is possible to find the coordinates of points by linear equation where terms containing (X_0, Y_0, Z_0) are transfer to the right part of equations. This leads to following expression for finding the coordinates:

$$\begin{aligned} &(c.r_{11} + x.r_{13}).X + (c.r_{21} + x.r_{23}).Y + (c.r_{31} + x.r_{33}).Z = \\ &\quad = (c.r_{11} + x.r_{13}).X_0 + (c.r_{21} + x.r_{23}).Y_0 + (c.r_{31} + x.r_{33}).Z_0 \\ &(c.r_{12} + x.r_{13}).X + (c.r_{22} + x.r_{23}).Y + (c.r_{32} + x.r_{33}).Z = \\ &\quad = (c.r_{12} + x.r_{13}).X_0 + (c.r_{22} + x.r_{23}).Y_0 + (c.r_{32} + x.r_{33}).Z_0 \end{aligned} \quad (9.9)$$

One method for linear space resection is suggested by Müller/Killian. This method is based on usage of four full control points. This is with one point more that than is necessary for a spatial resection with metric camera. Redundant information is not used only for improvement of accuracy but also for linearisation of non-linear problem.

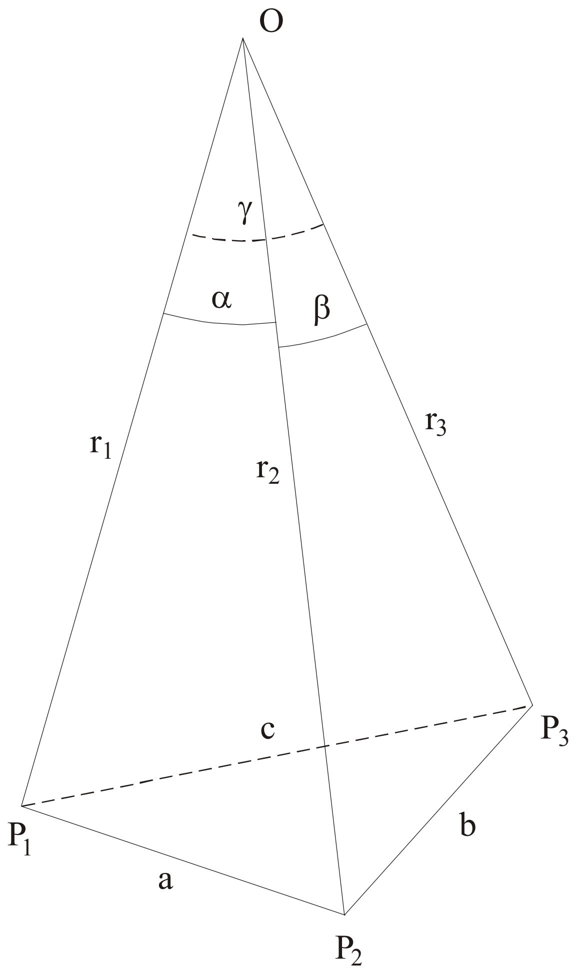


Figure 9.2. Spatial resection with three points

The known values are coordinates of points. Angles to the points can be measured by image coordinates and inner orientation parameters. The required values are coordinates of projection center of the camera (X_0, Y_0, Z_0). To solve the problem as the first step the lengths of tetrahedron must be computed. The following equations can be formulated:

$$
\begin{aligned}
r_1^2 + r_2^2 - 2.r_1.r_2.\cos\alpha = a^2 \\
r_2^2 + r_3^2 - 2.r_2.r_3.\cos\beta = b^2 \\
r_3^2 + r_1^2 - 2.r_3.r_1.\cos\gamma = c^2
\end{aligned}
\tag{9.12}
$$

After transformation the fourth-degree equation is formulated. The form of this equation is shown in Appendix 2

The solution of fourth degree equation can be found in different ways. One very effective suggestion is made by Killian. A fourth point P4 is introduced. It is possible to formulate second fourth-degree equation for variable v from points P1, P2 and P4. After elimination procedure an linear equation for v is reached. Thus the non-linear problem is transformed to linear and solved.
After finding the radii the coordinates of projection center can be found by intersection of three spheres with radii r1, r2 and r3 (spatial arc intersection). This also leads to fourth-degree equation.
The following solution of the problem is possible [following Kraus, K., 1997]. We project the cross point N of perpendicular from projection center O with three points plane (P_1, P_2, P_3).

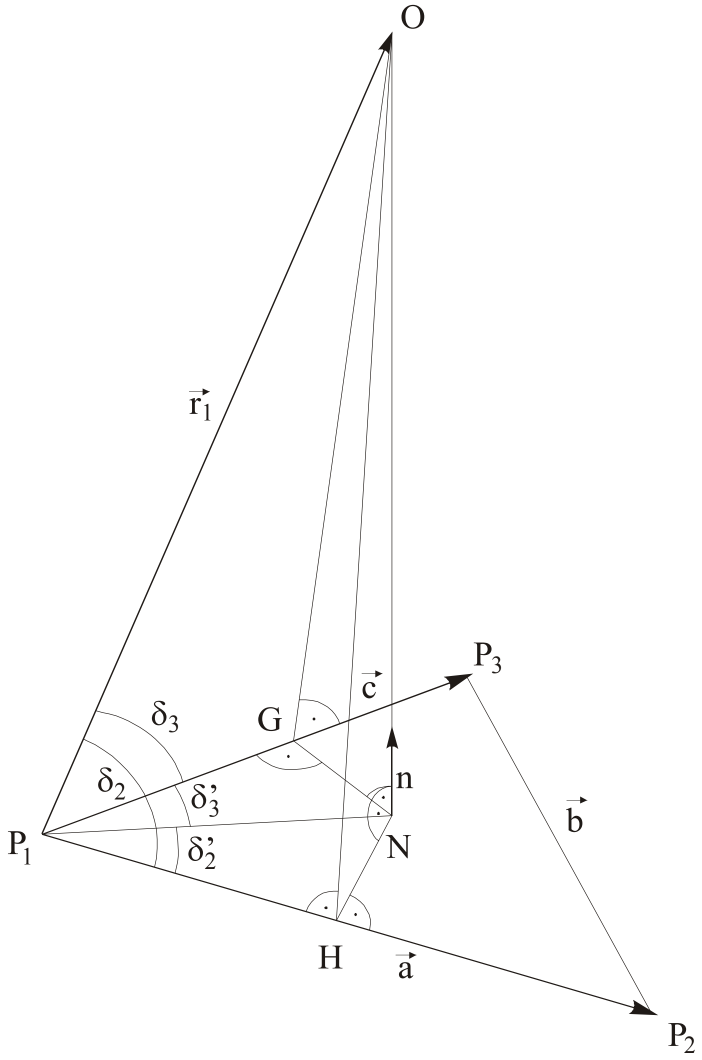


Figure 9.3. Intersection of projection vectors to three points

Three equations are formulated for calculation the increments of vector P1O are formulated.

The plane orthogonal to $\vec{a}$ and passing trough O is given by the scalar multiplication of two vectors $\vec{a}^t$ and $\vec{r}_1$. Accordingly to this we can construct the plane through O, that is orthogonal to $\vec{c}$.

The values of angles in the surrounding planes is calculated by the lengths of corresponding edges. For plane (P_1P_2O) these edges are a, r1, r2. For plane (P_1, P_3, O) these edges are b, r1, r3.
The third equation for coordinates of point N is derived from the fact that this point lies in the plane of (P_1, P_2, P_3). It is defined by orthogonal vector $\vec{n}$ to two of the vectors $\vec{a}$ and $\vec{b}$.
The solution of three equations for coordinate differences $\Delta X_N, \Delta Y_N$ and ΔZ_N allows to calculate the coordinates X_N, Y_N, Z_N

$$\begin{aligned} X_N &= X_1 + \Delta X_N \\ Y_N &= Y_1 + \Delta Y_N \\ Z_N &= Z_1 + \Delta Z_N \end{aligned} \qquad (9.13)$$

The length of perpendiclar $\overline{NO}$ is calculated by the relation

$$\overline{NO} = \sqrt{r_1^2 - \overline{P_1N}^2} = \sqrt{r_1^2 - (\Delta X_N^2 + \Delta Y_N^2 + \Delta Z_N^2)} \qquad (9.14)$$

Finally the coordinates of point O are calculated by the relation

$$\begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} X_N \\ Y_N \\ Z_N \end{bmatrix} + \overline{NO}.\frac{\boldsymbol{n}}{|\boldsymbol{n}|} \qquad (9.15)$$

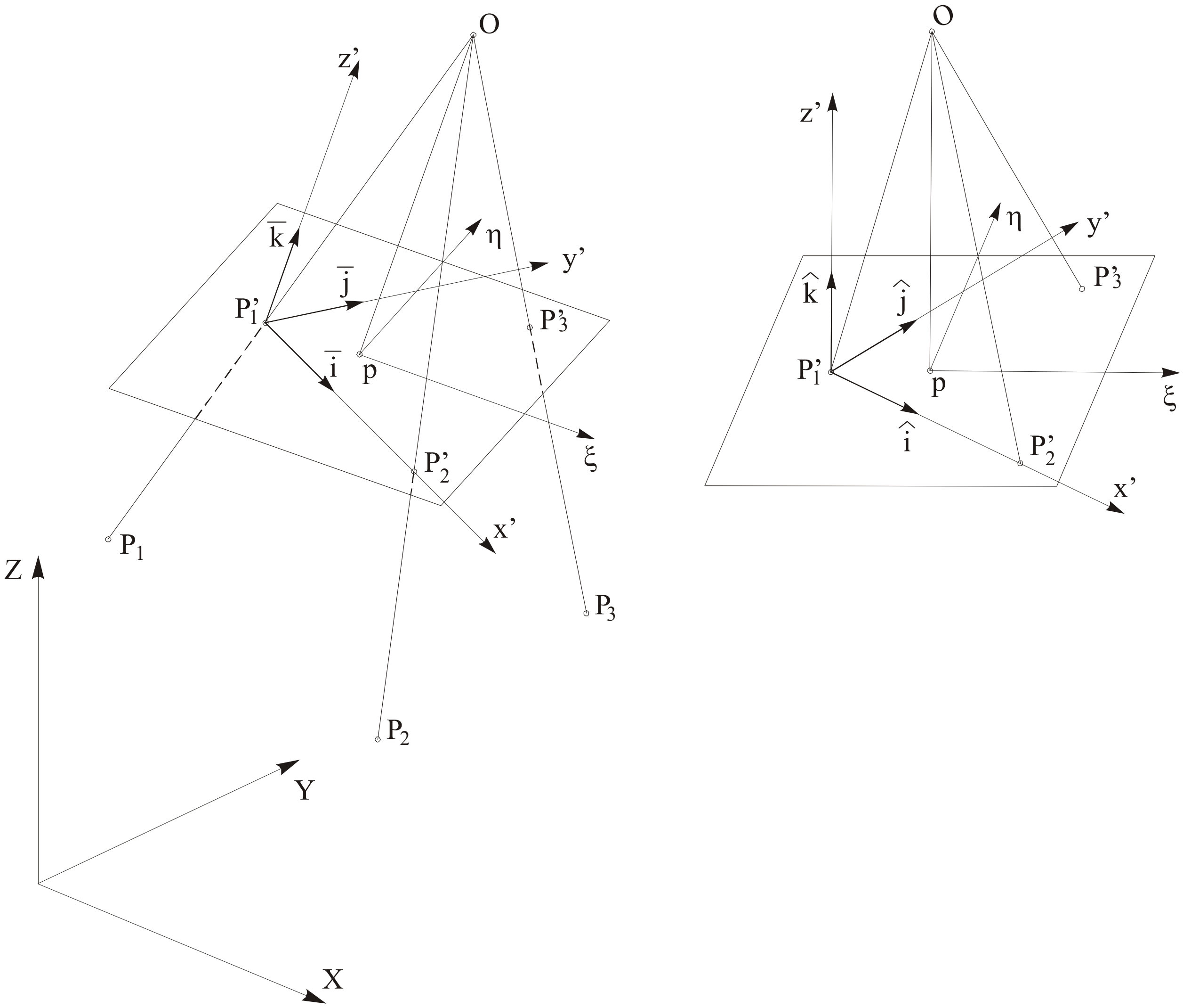
In the solution of spatial resection with three known points the six elements of outer orientation can be calculated from six linearised equations. The system of equations can be singular or ill-conditioned if three point lie on the circular cylinder with the projection center.
In the case of four points there is no dangerous cylinder. But in this case the dangerous construction is when four points lie on **horopter curve**, which lies on dangerous cylinder.

9.2.3. Rotation of bundle of rays

After calculation of the projection centers posistion the next step is determination of rotation between object coordinate system and image (camera) coordinate system. We must find the rotation matrix R, that transforms image coordinates x into object X (omitting scaling and translation).

$$X = R.x \qquad (9.16)$$

To solve the problem it is possible to define the image and object coordinate systems, connected with three given points.



9.4.Unit vectors in object and image coordinate systems

The calculation of unit vectors in image coordinate system is made on the base of measuring coordinates of image points. We form $(\xi_1, \eta_1, -c)$, $(\xi_2, \eta_2, -c)$ and $(\xi_3, \eta_3, -c)$. From these vectors we calculate unit vectors in image coordinate system - $\hat{i}, \hat{j}, \hat{k}$.

The calculation of unit vectors in object coordinate system is more complicated. For these purpose it is necessary to calculate the lengths of vectors to image points.

$$\overline{OP_i^{'}} = \sqrt{x_i^2 + y_i^2 + c^2} \qquad (9.17)$$

Detailed calculation of these vectors is presented in Appendix 4.

Two sets of vectors define rotation matrixes.

$$\overline{R} = \begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \end{bmatrix} \qquad \boldsymbol{X} = \boldsymbol{R}.\boldsymbol{x}' \qquad (9.18)$$

Matrix $\widehat{R}$ produces only rotation in the plane of the photo

$$\widehat{R} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \end{bmatrix} \qquad \boldsymbol{x} = \widehat{\boldsymbol{R}}.\boldsymbol{x}' \tag{9.19}$$

Combining the vector equations we find the total rotation matrix

$$\boldsymbol{X} = \overline{\boldsymbol{R}}.\widehat{\boldsymbol{R}}^{t}.\boldsymbol{x} = \boldsymbol{R}.\boldsymbol{x} \tag{9.20}$$

The presentation of transformation matrix is

$$\boldsymbol{R} = \overline{\boldsymbol{R}}.\widehat{\boldsymbol{R}}^{t} \tag{9.21}$$

This approach allows to determine more accurate the orientation of image in object coordinate system. However it requires three control points (with known coordinates to be measured in each photo. Some times such procedure is possible to be applied not to separate photo but for the whole photogrammetric model. In such case the calculations of models must be done and relative orientation of models to be performed too.

9.2.4. Sequence for resection and intersection

The procedure for determination of initial approximation runs in following sequence:

1.Numbering of all photographs

2.Numbering of all points in the set of photograph

3.Measurement of image coordinates of all points in all photographs

4.Storage of object coordinates of control points in a file

5.Transfer the control points to the file of object points

6.Finding photographs, in which at least four object points with known coordinates are imaged

7.Spatial resection of these photographs

8.Spatial intersection for all new points that lie in at least two photographs for which the spatial resection has been calculated

9.Transfer object coordinates of calculated points to the file of object points

10.Return to step 6 if there are non-oriented photos.

11.After calculation of all points the procedures stops

In some programs it is possible to use relative orientation for generation of models from photos where there are not enough points with known coordinates.

9.2.5.Initial approximation for spatial transformation

There are some case more often in close range photogrammetry when there are data for spatial 3D model.

It can be for example:

relatively oriented photogrammetric models;
digital object models (automobile body, building model from plans) etc.
tachymeter model in local Cartesian coordinates from polar measurements;
features defined in three dimensional coordinate system;
projection centers within flight strips, determined by means of GPS measurements;
The calculation is based on usage of equation of spatial similarity transformation

$$X = X_0 + m.R_{\Omega\Phi K} x$$
$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (9.22)$$

Space similarity transformation is defined by 7 parameters – coordinates of the center of coordinate system, scale and three rotation angles. This approach leads to non-linearity solution. If for orientation are used four homologues points with corresponding coordinates it is possible to calculate the 12 unknowns of transformation. After that the scale factor can be computed from relation

$$m = \frac{\sqrt{\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}^2}}{3} \quad (9.23)$$

When the scaling factor is calculated all coefficients of rotational matrix are divided to m.

$$r_{ij} = \frac{a_{ij}}{m} \qquad \text{for} \quad i = 1 \div 3, \; j = 1 \div 3 \quad (9.24)$$

From the relations for coefficients and angles the angles values can be calculated.

Except the common points it is possible to use common straight lines or spatial lines to be used for tie elements between images. The analytical formulas for these lines are used and their parameters are added as unknowns to the normal system of equations. In some cases the line can be represented as intersection of two polynomial surfaces.

$$z_0 = c_{11}.x.y$$
$$y_0 = b_{30}.x^3 + b_{40}.x^4 + b_{21}.x^2.z \quad (9.25)$$

9.3.Solution with colinearity conditions

When the configuration of photos is arbitrary and there are more then two measurements in images for image point the most appropriate method is based on usage of colinearity equations.

$$x = x_p - c\frac{r_{11}.(X - X_0) + r_{21}.(Y - Y_0) + r_{31}.(Z - Z_0)}{r_{13}.(X - X_0) + r_{23}.(Y - Y_0) + r_{33}.(Z - Z_0)}$$
$$y = y_p - c\frac{r_{12}.(X - X_0) + r_{22}.(Y - Y_0) + r_{32}.(Z - Z_0)}{r_{13}.(X - X_0) + r_{23}.(Y - Y_0) + r_{33}.(Z - Z_0)} \quad (9.26)$$

The colinearity equations are observation equations because the calculated values are directly measured coordinates. From them the correction equations are produced

$$v_x + x_m = x_p - c\frac{r_{11}.(X - X_0) + r_{21}.(Y - Y_0) + r_{31}.(Z - Z_0)}{r_{13}.(X - X_0) + r_{23}.(Y - Y_0) + r_{33}.(Z - Z_0)}$$
$$v_y + y_m = y_p - c\frac{r_{12}.(X - X_0) + r_{22}.(Y - Y_0) + r_{32}.(Z - Z_0)}{r_{13}.(X - X_0) + r_{23}.(Y - Y_0) + r_{33}.(Z - Z_0)} \quad (9.27)$$

They can be transformed into the form

$$v_x = x_p - c\frac{r_{11}.(X - X_0) + r_{21}.(Y - Y_0) + r_{31}.(Z - Z_0)}{r_{13}.(X - X_0) + r_{23}.(Y - Y_0) + r_{33}.(Z - Z_0)} - x_m$$
$$v_y = y_p - c\frac{r_{12}.(X - X_0) + r_{22}.(Y - Y_0) + r_{32}.(Z - Z_0)}{r_{13}.(X - X_0) + r_{23}.(Y - Y_0) + r_{33}.(Z - Z_0)} - y_m \quad (9.28)$$

The correction equations are non-linear. They can be linearised by differentiating accordingly to the unknowns

$$v_x = b_{11}\delta X_0 + b_{12}.\delta Y_0 + b_{13}\delta Z_0 + b_{14}\delta\omega + b_{15}\delta\varphi + b_{16}\delta\kappa$$
$$+b_{17}\delta X + b_{18}\delta Y + b_{19}\delta Z + b_{1,10}\delta x_p + b_{1,12}\delta c - l_x$$
$$v_y = b_{21}\delta X_0 + b_{22}.\delta Y_0 + b_{23}\delta Z_0 + b_{24}\delta\omega + b_{25}\delta\varphi + b_{26}\delta\kappa$$
$$+b_{27}\delta X + b_{28}\delta Y + b_{29}\delta Z + b_{2,11}\delta y_p + b_{2,12}\delta c - l_y \quad (9.29)$$

where

$$l_x = x_m - x$$
$$l_y = y_m - y \quad (9.30)$$

In matrix form the correction equations are presented in the form

$$\underset{(m\times 1)}{\boldsymbol{V}} = \underset{(m\times n)}{\boldsymbol{B}} . \underset{(n\times 1)}{\boldsymbol{x}} - \underset{(m\times 1)}{\boldsymbol{l}} \quad (9.31)$$

where m – number of observations,

n – number of unknowns.

The least square adjustment can be applied if number of observation is greater then the number of unknowns m>n.

The conditions for correct application of least square method are:

a) the normal distribution of residuals vi with zero mean value and known variance σ;

b) mutually independent observations

Forming and solution of normal equation system is given in Appendix 5.

The solution of normal system is given by equation

$$\boldsymbol{x} = \boldsymbol{N}^{-1}\boldsymbol{L} \tag{9.32}$$

The variance-covariance matrix for x is obtained by

$$\sigma_x = \sigma_0^2 \boldsymbol{N}^{-1} \tag{9.33}$$

The application of colinearity equations and adjustment by observation equations gives the solutions in all cases of combinations of measurements and photos. It is necessary that the number of observations to new points to be greater or equal of two.

Appendixes

Appendix 1

The necessary number of equations k is calculated by the relation

$$k = 3.k_p + 3.n_p^n \tag{9.34}$$

where k_p is number of photos and n_p^n is the number of new points

The possible number of equations is defined by formula

$$k_e = 2.n_{p1} + 4.n_{p2} + 6.n_{p3} + ... = 2.[n_p + (n_p - n_{p1}) + (n_p - n_{p1} - n_{p2}) + (n_p - n_{p1} - n_{p2} - n_{p3}) + ...] = \sum_{m=1}^{m_{\max}} 2.m.n_{pm} \tag{9.35}$$

where m is the number of occurrence of point and n_{pm} is the number of points with occurrence of order m. It is evident that if there are only points image only in one photo it is not possible to find the solution. But if points are measured in two or more photos then it is possible to select enough equations. The number of equations from new points from which we can determine projection centers is decreased with three. So the following relation must be satisfied.

$$\begin{aligned} 3.k_p + 3.n_p^n &\le \sum_{m=1}^{m_{\max}} 2.m.(n_{pm}^c + n_{pm}^n) \\ 3.k_p &\le \sum_{m=1}^{m_{\max}} [2.m.n_{pm}^c + (2.m.n_{pm}^n - 3.n_p^n)] \end{aligned} \tag{9.36}$$

The above equation can be presented in the following form:

$$3.k_p \le 2.n_{p1}^c + 4.n_{p2}^c + 6.n_{p3}^c + (4.n_{p2}^n - 3) + (6.n_{p3}^n - 3) + ... \tag{9.37}$$

Appendix 2

The following new unknown is introduced.

$$v = \frac{r_2}{r_1} \tag{9.38}$$

Finally the equation for v has the form

$$\begin{aligned}
&[1 - 2.A + B + 4.C.\sin^2\beta].v^4 \\
&+4.[-\cos\beta.\cos\gamma + A.(\cos\beta.\cos\gamma + \cos\alpha) - B.\cos\alpha - 2.C.\sin^2\beta.\cos\alpha].v^3 \\
&+2[1 + 2(\cos^2\beta - \sin^2\gamma) - 2A(1 + 2\cos\beta\cos\alpha\cos\gamma) \\
&\qquad + B(1 + 2\cos^2\alpha) + \frac{2}{a^2}(b^2\sin^2\gamma + c^2\sin^2\beta)]v^2 \\
&+4.[-\cos\beta.\cos\gamma + A.(\cos\beta.\cos\gamma + \cos\alpha) - B.\cos\alpha - 2.D.\sin^2\gamma.\cos\alpha].v \\
&[1 - 2.A + B + 4.D.\sin^2\gamma] \ = \ 0
\end{aligned} \tag{9.39}$$

where A, B, C and D are represented by following expressions

$$A = \frac{b^2 + c^2}{a^2}\ , \qquad B = \left(\frac{b^2 - c^2}{a^2}\right)^2\ , \qquad C = \frac{c^2}{a^2}\ , \qquad D = \frac{b^2}{a^2} \tag{9.40}$$

Appendix 3

The plane orthogonal to $\vec{a}$ and passing trough O is given by the scalar multiplication of two vectors $\vec{a}^{\,t}$ and $\vec{r_1}$. This gives the relation

$$\boldsymbol{a}^t.\boldsymbol{r_1} = \Delta X_2.\Delta X_0 + \Delta Y_2.\Delta Y_0 + \Delta Z_2.\Delta Z_0 = \overline{P_1P_2}.r_1.\cos\delta_2 = a.r_1.\cos\delta_2 = \overline{P_1H} \tag{9.41}$$

where increments of coordinates are relatively to point P1.

For the foot point N, that lies in the same plane we have the equation

$$\Delta X_2.\Delta X_N + \Delta Y_2.\Delta Y_N + \Delta Z_2.\Delta Z_N = \overline{P_1P_2}.\overline{P_1N}.\cos\delta_2' = \overline{P_1H} = a.r_1.\cos\delta_2 \tag{9.42}$$

Accordingly to this we can construct the plane through O, that is orthogonal to $\vec{c}$.

$$\boldsymbol{c}^t.\boldsymbol{r_1} = \Delta X_3.\Delta X_0 + \Delta Y_3.\Delta Y_0 + \Delta Z_3.\Delta Z_0 = \overline{P_1P_3}.r_1.\cos\delta_2 = c.r_1.\cos\delta_3 \tag{9.43}$$

The equation for the point N in this plane takes the form

$$\Delta X_3.\Delta X_N + \Delta Y_3.\Delta Y_N + \Delta Z_3.\Delta Z_N = \overline{P_1P_3}.\overline{P_1N}.\cos\delta_3' = \overline{P_1G} = c.r_1.\cos\delta_3 \tag{9.44}$$

In the above equations the values of angles in the surrounding planes is calculated by the lengths of corresponding edges. For plane (P_1P_2O) these edges are a, r1, r2. For plane (P_1,P_3,O) these edges are b, r1, r3.

The third equation for coordinates of point N is derived from the fact that this point lies in the plane of (P_1,P_2,P_3). It is defined by orthogonal vector $\vec{n}$ to two of the vectors $\vec{a}$ and $\vec{b}$.

$$\boldsymbol{n} = \boldsymbol{a} \times \boldsymbol{b}$$

$$\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = \begin{bmatrix} a_y.b_z - a_z.b_y \\ a_z.b_x - a_x.b_z \\ a_x.b_y - a_y.b_x \end{bmatrix} \tag{9.45}$$

The equation for coordinates of N in the plane of three points is

$$n_x.\Delta X_N + n_y.\Delta Y_N + n_z.\Delta Z_N = 0 \tag{9.46}$$

The solution of three equations for coordinate differences $\Delta X_N, \Delta Y_N$ and ΔZ_N allows to calculate the coordinates X_N, Y_N, Z_N

Appendix 4

The unit vector ***i*** is defined to lie on the line $P_1^{'}, P_2^{'}$.

$$\bar{i} = \frac{\overline{P_1^{'}P_2^{'}}}{\left\| P_1^{'}P_2^{'} \right\|} \tag{9.47}$$

Vector k is orthogonal to the image plane, so we have

$$\bar{k} = \frac{\overline{P_1^{'}P_2^{'}} \times \overline{P_1^{'}P_3^{'}}}{\left\| \overline{P_1^{'}P_2^{'}} \times \overline{P_1^{'}P_3^{'}} \right\|} \tag{9.48}$$

Finally it is possible to find vector j using the fact that we have orthogonal coordinate system.

$$\bar{j} = \bar{k} \times \bar{i} \tag{9.49}$$

In the similar way it is possible to calculate unit vectors in the image coordinate system. Two sets of vectors define rotation matrixes.

Appendix 5

The normal equation system is formed by relations

$$\left(\boldsymbol{B}^t \boldsymbol{W} \boldsymbol{B}\right) \boldsymbol{x} = \boldsymbol{B}^t \boldsymbol{W} \boldsymbol{l} \tag{9.50}$$

If define

$$\boldsymbol{N} = \boldsymbol{B}^t\boldsymbol{W}\boldsymbol{B} \qquad \boldsymbol{L} = \boldsymbol{B}^t\boldsymbol{W}\boldsymbol{l} \tag{9.51}$$

Then normal system is presented as

$$\boldsymbol{N}.\boldsymbol{x} = \boldsymbol{L} \tag{9.52}$$

The matrix W is weight matrix and in the case uncorrelated measurement has the form

$$W = \sigma_0^2 \begin{bmatrix} \delta x_1^2 & & \\ & \sigma x_2^2 & \\ & & \sigma x_m^2 \end{bmatrix}^{-1} \tag{9.53}$$

The solution of normal system is given by equation

$$\boldsymbol{x} = \boldsymbol{N}^{-1}\boldsymbol{L} \tag{9.54}$$

The variance-covariance matrix for x is obtained by

$$\sigma_x = \sigma_0^2 \boldsymbol{N}^{-1} \tag{9.55}$$

where σ_0 is unit weight variance and it is calculated by

$$\sigma_0^2 = \frac{\sum_{j=1}^{m} v_j^2}{m-n} \tag{9.56}$$